Complex Analysis

Math 214 Spring 2004 — Fowler 112 MWF 3:30pm - 4:25pm

 — http://faculty.oxy.edu/ron/math/312/04/

Class 9: **Monday February 9**

SUMMARY The Cauchy-Riemann Equations
CURRENT READING Saff & Snider, §2.3
HOMEWORK Saff & Snider, Section 2.4 # 1, 2, 4, 5, 6 **Extra Credit: # 15**

GroupWork

Given $g(z) = z^2 + z + i$ and $f(z) = \dfrac{1}{z}$

$g'(z) =$

$f'(z) =$

$[g(z)f(z)]' =$

$[g(z)/f(z)]' =$

Cauchy-Riemann Equations
We shall derive the Cauchy-Riemann equations by looking at the definition of the derivative of a function $f(z) = u(x, y) + iv(x, y)$ at the point z_0.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x + i\Delta y} \end{aligned}$$

We shall do this limit twice, once letting $\Delta z \to 0$ horizontally and the other time letting $\Delta z \to 0$ vertically

This shows us that the existence of $f'(z_0)$ implies the Cauchy-Riemann equations are satisfied at this point (and at every point in a neighborhood of z_0).
This is true since if $f'(z_0)$ exists then f is **analytic** "at" this point.

The Cauchy-Riemann Equations
Given a function $f(z) = u(x, y) + iv(x, y)$ the corresponding Cauchy-Riemann Equations are

$$u_x = v_y, u_y = -v_x$$

$$\text{ANALYTICITY} \Rightarrow \text{C.R.E.}$$

To make satisfying the CRE a *sufficient* condition one needs the added condition that the first derivatives of u and v are continuous. If both these conditions are true and f is defined on an open set, then f is analytic on the open set.

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -i(u_y(x_0, y_0) + iv_y(x_0, y_0))$$

$$\text{ANALYTICITY} \Longleftrightarrow \text{C.R.E.} + \text{Continuity of } u_x, u_y, v_x, v_y$$

EXAMPLE

Show that $f(z) = \overline{z}$ is **not analytic** anywhere in the complex plane. You can do this in two ways:

1:

2:

GroupWork

Show that the function $f(z) = 1/z$ is analytic on the set $z \neq 0$. To do that you will have to answer the following questions:

- What is its domain of definition? Is this an open set?
- What are its component functions? Are their partial derivatives continuous?
- Do they satisfy the CRE?
- Is it analytic? On what set? Is this set open or closed?