Complex Analysis

Math 214 Spring 2004 — Fowler 112 MWF 3:30pm - 4:25pm

 — http://faculty.oxy.edu/ron/math/312/04/

Class 5: **Friday January 30**

SUMMARY Point Sets in the Complex Plane
CURRENT READING Saff & Snider, §1.6
HOMEWORK Saff & Snider, Section 1.5 # 3, 4, 5, 6, 10, 11, 15 **Extra Credit: #21** and Section 1.6 # 2,3,4,5,6,7,8

Any collection of points in the complex plane is called a *two-dimensional* point set, and each point is called a *member* or *element* of the set. Here are some fundamental definitions describing planar point sets.

Definitions

NEIGHBORHOOD
A *delta* or δ *neighborhood* of a point z_0 is the set of all points z such that $|z - z_0| < \delta$ where δ is any given positive (real) number.

DELETED NEIGHBORHOOD
A *deleted* δ *neighborhood* of z_0 is a neighborhood of z_0 in which the point z_0 is omitted, i.e. $0 < |z - z_0| < \delta$

LIMIT POINT
A point z_0 is called a *limit point, cluster point* or a *point of accumulation* of a point set S if every deleted δ neighborhood of z_0 contains points of S. Since δ can be any positive number, it follows that S must have infinitely many points. Note that z_0 may or may not belong to the set S.

INTERIOR POINT
A point z_0 is called an *interior point* of a set S if we can find a neighborhood of z_0 all of whose points belong to S.

BOUNDARY POINT
If every δ neighborhood of z_0 conrains points belonging to S and also points not belonging to S, then z_0 is called a *boundary point.*

EXTERIOR POINT
If a point is not a an interior point or a boundary point of S then it is called an *exterior point* of S.

OPEN SET
An *open set* is a set which consists only of interior points. For example, the set of points $|z| < 1$ is an open set.

CLOSED SET
A set S is said to be closed if every limit point of S belongs to S, i.e. if S contains all of its limit points. For example, the set of all points z such that $|z| \leq 1$ is a closed set.

BOUNDED SET
A set S is called *bounded* if we can find a constant M such that $z < M$ for every point in S. An *unbounded set* is one which is not bounded. A set which is both closed and bounded is sometimes called *compact.*

CONNECTED SET
An open set S is said to be *connected* if any two points of the set can be joined by a path consisting of straight line segments (i.e. a *polygonal* path) all points which are in S.

DOMAIN or OPEN REGION
An open connected set is called an *open region* or *domain.*

CLOSURE
If to a set S we add all the limit points of S, the new set is called the *closure* of S and is a closed set.
CLOSED REGION
The closure of an open region or domain is called a *closed region.*
REGION
If to an open region we add some, all or none of its limit points we obtain a set called a *region.* If all the limit points are added the region is *closed*; if none are added the region is *open.* Usually if the word *region* is used without qualifying it with an adjective, it is referring to an *open region* or *domain.*

Notes

Yes, closed sets can be connected. Open connected sets are more interesting because they are also called **domains** or open regions. If a set is closed and connected it's called a closed region.

If a set does not have any limit points, such as the set consisting of the point $\{0\}$, then it is **closed**. [It contains all its limit points (it just doesn't have any limit points).]

Remember, if a set contains all its boundary points (marked by solid line), it is **closed.**
If a set contains none of its boundary points (marked by dashed line), it is **open.**

Also, *some sets can be both open and closed.* An example is the set $\mathcal{C}$ (the Complex Plane). It has no boundary points. Thus $\mathcal{C}$ is closed since it contains all of its boundary points (doesn't have any) and $\mathcal{C}$ is open since it doesn't contain any of its boundary points (doesn't have any).

Also, *some sets can be neither open or closed.* The set $0 < |z| \leq 1$ has two boundaries (the set $|z| = 1$ and the point $z = 0$). It contains the first boundary ($|z| = 1$), so it is not open, but it does not contain the boundary point $z = 0$ so it is not closed. $z = 0$ is also a limit point for this set which is not in the set, so this is another reason the set is not closed.

EXAMPLES

Consider the following point sets and, using as many of the previous definitions as you can, fully describe these examples.

$\mathbf{|z - i| \geq 3}$ $\qquad$ $\mathbf{-1 < \text{Im } z < 3}$ $\qquad$ $\mathbf{1 < |z|}$

$\mathbf{|z + 1| \leq 1 \cap |z - 1| \leq 1}$ $\qquad$ $\mathbf{|z + 1| \leq .5 \cup |z - 1| \leq .5}$ $\qquad$ $\mathbf{\text{Re } z = 0}$