TEST 1: Friday, March 5, 2004

Directions: Read all 3 problems first before answering any. Notice the HINTS on each problem. You may have access to any notes or the textbook. This is a one hour test. You have 90 minutes to complete it.

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1. [40 pts. total] **Mapping.** We want to understand the implications on the location and orientation of the branch cut of the Principal Branch of the Complex Logarithm function when the argument changes from $\log(z)$ to $\log(Az + B)$ where $A$ and $B$ are complex numbers.

**HINT:** The principal branch cut is at $D = \{z : \Re(z) \leq 0 \cap \Im(z) = 0\}$.

(a) [5 pts/ Sketch the location and orientation of the branch cut of $\log(z)$ under the mapping $w = f_1(z) = z + z_0$ where $z_0 \in \mathcal{C}$.]

(b) [5 pts/ Sketch the location and orientation of the branch cut of $\log(z)$ under the mapping $w = f_2(z) = e^{i\theta}z$ where $\theta \in \mathcal{R}$.

(c) [5 pts/ Sketch the location and orientation of the branch cut of $\log(z)$ under the mapping $w = iz + 1$ where $\theta \in \mathcal{R}$.]

(d) [5 pts/ Sketch the location and orientation of the branch cut of $\log(iz + 1)$. [HINT: think about how part (c) and (d) are related questions, but different!]}
(e) [10 pts] Find a function \( \log (Az + B) \) which has its branch cut located at
\[ D = \{ z = x + iy : y = x \cap x \geq 1 \} \]

(f) [10 pts] Is it possible to find a single-valued branch of \( \log(z) \), i.e. \( \mathcal{L}_\alpha(z) \) which has its branch cut at the same exact location and orientation as \( D \) from part (e)? **Why, or why not?** If possible, write down a function involving \( \mathcal{L}_\alpha \) which has the same branch cut \( D \).
2. [30 pts.] Algebra of Complex Numbers.
   Consider \( \cos(z) = b \) where \( b \in \mathbb{C} \). Therefore, \( z = \cos^{-1}(b) = \arccos(b) \)
   
   **HINT:** \( \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \).

   (a) [10 pts] Show that \( z = \arccos(b) = -i \log(b \pm \sqrt{b^2 - 1}) \)

   (b) [10 pts] When \( b = 0 \), use information from part (a) to help you evaluate
   \( z = \arccos(0) \). **Indicate the location of your solutions in the Complex Plane.**

   (c) [10 pts] When \( b = 2 \), use information from part (a) to help you evaluate
   \( z = \arccos(2) \) **Indicate the location of your solutions in the Complex Plane.**
3. [30 pts. total] **Cauchy-Riemann Equations, Harmonic Conjugates.** Consider an analytic function \( f(z) = u(x, y) + iv(x, y) \). We want to show that the set of implicitly-defined curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) are orthogonal to each other at their point of intersection. Note that \( c_1 \) and \( c_2 \) are real constants. **HINT:** two curves are perpendicular whenever the product of their slopes equals -1.

(a) [10 pts] For \( f(z) = Az + B \) where \( A \) and \( B \) are complex constants, find \( u(x, y) \) and \( v(x, y) \). Show that the slopes of the implicit curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) are perpendicular to each other whenever they intersect.

(b) [10 pts] For \( f(z) = z^2 \) find \( u(x, y) \) and \( v(x, y) \) and show that the slopes of the implicit curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) are perpendicular whenever they intersect.

(c) [10 pts] Building on your answers in (a) and (b), use implicit differentiation and the Cauchy Riemann Equations to prove the general principle that for an analytic function \( f(z) = u(x, y) + iv(x, y) \) the family of curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) are orthogonal (i.e. perpendicular at a general point in the \( xy \)-plane).