Final Exam

Math 224 **Multivariable Calculus**

Wednesday, May 5, 2004: 8:30–11:30am Prof. R. Buckmire

Name: ____________________

Directions: Read *all* problems first before answering any of them. There are TEN (10) problems on ELEVEN (11) pages.

This exam is a limited-notes, closed-book, test. You may use a calculator and bring in one 8.5 x 11 inch sheet of paper.

You must include ALL relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answer from your "scratch work.'

No.	Score	Maximum
1.		**20**
2.		**20**
3.		**20**
4.		**20**
5.		**20**
6		**20**
7.		**20**
8.		**20**
9.		**20**
10.		**20**
TOTAL		**200**

1. *[20 points total.]* **Vector Operations, Equations of Lines**
Consider the two vectors $\vec{\mathbf{a}} = (1, -2, 1)$ and $\vec{\mathbf{b}} = (-2, 1, 1)$.
(a) (*5 points.*) Compute **a** - 3**b**.

(b) (*5 points.*) Find the coordinates of the midpoint between **a** and **b**.

(c) (*5 points.*) Write down the vector equation of the line joining **a** and **b**.

(d) (*5 points.*) Is the point $(-5, 4, 1)$ on the line joining **a** and **b**? How do you know? **Explain your answer!**

2. *[20 points total.]* **Equations of Planes and Vector Products.**
Consider the two vectors $\vec{a} = (1, -2, 1)$ and $\vec{b} = (-2, 1, 1)$.
(a) (*5 points.*) Compute $\vec{a} \cdot \vec{b}$.

(b) (*5 points.*) Compute $\vec{a} \times \vec{b}$.

(c) (*10 points.*) Find the equation of the plane that contains $\vec{a}$ and $\vec{b}$. (HINT: and their midpoint!)

3. *[20 points total.]* **Multivariable Limits.**
Evaluate the following limits.

(b) *(5 points.)* $\lim_{(x,y)\to(0,0)} \dfrac{x^3 - y^3}{x^3 + y^3}$

(b) *(5 points.)* $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

(c) *(5 points.)* $\lim_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2}$

(d) *(5 points.)* $\lim_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{xy}$

4. *[20 points total.]* **Partial Derivatives.**
Consider the function $f(x, y) = x^y$.
(a) (*5 points.*) Show that $f_{xy} = f_{yx}$ for this function.

(b) (*5 points.*) Compute $\vec{\nabla} f$ at (2,1).

(c) (*10 points.*) Use your answers above to approximate the value of $f(2.01, .97)$

5. *[20 points total.]* **Multivariable Chain Rule.**

Consider $\vec{f}(x,y) = \begin{bmatrix} x^2+y^2 \\ x^2-y^2 \end{bmatrix}$ and $\vec{g}(u,v) = \begin{bmatrix} uv \\ u+v \end{bmatrix}$.

(a) (*5 points.*) Compute $\vec{g} \circ \vec{f}$.

(b) (*10 points.*) Find the derivative matrix (jacobian) of $\vec{f}$ and $\vec{g}$, i.e. $\dfrac{\partial \vec{f}}{\partial(x,y)}$ and $\dfrac{\partial \vec{g}}{\partial(u,v)}$.

(c) (*5 points.*) Compute $(\vec{g} \circ \vec{f})'$

6. *[20 points total.]* **Multivariable Optimization, Lagrange Multipliers**
Find the points on the ellipse $5x^2 - 6xy + 5y^2 = 4$ which are closest to and furthest from the origin. (HINT: Optimize the square of the distance!)

7. *[20 points total.]* **Iterated Integration.**

(a) (*5 points.*) $\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\ dy\ dx$

(b) (*5 points.*) $\int_0^1 \int_0^2 (x+y)^2\ dx\ dy$

(c) (*10 points.*) $\int_{-1}^1 \int_{y^2}^1 \int_0^{1-x}\ dz\ dx\ dy$

8. *[20 points total.]* **Multiple Integration.**

(a) (*10 points.*) Evaluate $\displaystyle\iiint_S \frac{dx\, dy\, dz}{(x^2+y^2+z^2)^{3/2}}$ where S is the solid bounded by the spheres
$x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ and $a>b>0$.

(b) (*5 points.*) Consider the region R bounded by the graphs of $y=x^3$ and $x=y^2$. Evaluate the integral $\displaystyle\iint_R x\, dx\, dy$

(c) (*5 points.*) Compute $\displaystyle\int_D \cos(x^2+y^2)\, dx\, dy$ where D is the disk of radius $\sqrt{\pi/2}$ centered at (0,0).

9. *[20 points total.]* **Gradient Fields, Div, Grad and Curl.**
Consider the vector field $\vec{F}$ which is the gradient of the Newtonian potential function $f(\vec{x}) = -|\vec{x}|^{-1}$ for non-zero $\vec{x}$ in $\mathbb{R}^3$.
(a) (*10 points.*) Show that $\vec{F}$ is indeed a vector field.

(b) (*10 points.*) Find the work done (by evaluating $\int_\Gamma \vec{F} \cdot d\vec{x}$) in moving a particle from $(1, 1, 1)$ to $(-2, -2, -2)$ along a smooth curve Γ lying in the domain of $\vec{F}$.

10. *[20 points total.]* **Green's Theorem.**

By evaluating the line integral $\frac{1}{2}\oint_{\Gamma} x\ dy - y\ dx$ and using Green's Theorem, show that the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the minor and major axes of the ellipse and Γ is the closed path in $\mathbb{R}^2$ traced out by the ellipse.