Report on Test 2

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Point Distribution (N=16)

Range	85+	80+	76+	68+	65+	60+	54+	50+	50-	40-
Grade	A	A-	B+	B	B-	C+	C	C-	D	F
Frequency	4	3	0	2	1	2	1	0	3	0

Comments Clearly this exam was more difficult than the first exam. The average was 9 points lower, to 71.6. Almost 1/2 (7 out of 16) of the class earned an A or A- on this exam. A pretty good performance overall! The low score was 46 and the high score was 98. The median score was 73.

#1 Chain Rule, Implicit Function Theorem. The hardest question was first. It's a concept-based question about the heart of the Chain Rule. When you have a function depending on some variables, regardless of what they are and you are asked to take a partial derivative with respect to a DIFFERENT variable, you must partially differentiate with respect to those variables. Suppose $f(\heartsuit, \diamondsuit, \clubsuit, \spadesuit)$ and you're asked to find $\frac{\partial f}{\partial x}$ then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \heartsuit}\frac{\partial \heartsuit}{\partial x} + \frac{\partial f}{\partial \diamondsuit}\frac{\partial \diamondsuit}{\partial x} + \frac{\partial f}{\partial \clubsuit}\frac{\partial \clubsuit}{\partial x} + \frac{\partial f}{\partial \spadesuit}\frac{\partial \spadesuit}{\partial x}$. Then to simplify this you just need to know how (or if!) $\heartsuit, \diamondsuit, \clubsuit, \spadesuit$ depend on x. (1b.) If you solve the expressions you get in part (1a.) using the Chain Rule, for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ you get the same results that you would get if you apply the Implicit Function Theorem on $F(x, y, z) = 0$ where $z = f(x, y)$. The main point of the IFT is figuring out which variables play the role of the dependent variable $\vec{y}$ and which play the role of the independent variable $\vec{x}$ in the expression $\vec{F}(\vec{x}, \vec{y}) = \vec{0}$. Since $z = f(x, y)$ which corresponds to $\vec{y} = \vec{G}(\vec{x})$ and we know $\vec{y} \leftrightarrow z$ and $\vec{x} \leftrightarrow (x, y)$. Also $\vec{G} \leftrightarrow f$ and $\vec{F} \leftrightarrow F$. Then you can use the IFT formula $G'(\vec{x}) = -(F_{\vec{y}})^{-1}F_{\vec{x}}$ which in this case becomes $(f_x \quad f_y) = (F_z)^{-1}(F_x \quad F_y)$.

#2 Partial Differentiation, Gradient Operator. This is a concrete example of Question 1. $F(x, y, z) = \cos(x+y+z) - xyz = 0$ is an explicit function and $f(x, y)$ is an implict function that can not be obtained explicitly since the function F can not be arranged to make z equal some function of x and y that we can write down. (2b). The Implicit Function Theorem allows us to differentiate $z = f(x, y)$ with respect to its variables even though we don't know f! So the two dimensional vector $\vec{\nabla} f = (f_x, f_y) = (-\frac{F_x}{F_z}, -\frac{F_x}{F_z})$. However F is explicit so you can just take its gradient (which is 3 dimensional) $\vec{\nabla} F = (F_x, F_y, F_z) = (-\sin(x+y+z)-yz, -\sin(x+y+z)-xz, -\sin(x+y+z)-xy)$

#3 Iterated Integration. (3a.) Note the re-appearance of $\cos(x + y + z) - xyz$. This is clearly the function of $F(x, y, z) = 0$ from Question 2 (There's always themes on my tests!) But it also serves as a useful function to test whether the idea that in iterated integration the other variables are treated as constants has been completely understood. Thanks to the symmetric -1 to 1 limits on x the xyz term will integrate to zero, though the trig terms will propagate like bunnies. Basically if you got that far you got 9/10 points. (3b.) If you integrate in this order you end up with $\int_1^2 \frac{\ln x}{x} dx$ which you need u-substitution to convert into $\int_0^{\ln 2} u \, du = \frac{1}{2}(\ln 2)^2$. If you don't like that integral, Remember Fubini! You can convert it into $\int_0^{\ln 2}\int_{e^y}^2 \frac{1}{x}\, dx\, dy = \int_0^{\ln 2} \ln(2) - \ln(e^y)\, dy = \int_0^{\ln 2} \ln(2) - y\, dy = \ln(2)\ln(2) - \frac{(\ln(2))^2}{2} = \frac{(\ln(2))^2}{2}$.

#4 Multiple Integration. Once you see the words "circle of radius..." you should be thinking polar coordinates (it's not cylindrical or spherical coordinates because the integral is $\int\int\, dA$ and not $\int\int\int dV$.) If you write it in cartesian coordinates you should get $\displaystyle\int_0^4 \int_0^{\sqrt{4^2-x^2}} ye^x\; dy\; dx$. In polar coordinates this becomes $\displaystyle\int_0^{\pi/2}\int_0^4 r\sin\theta e^{r\cos\theta}\; r\; dr\; d\theta$. This is a hard integral to do in this order, so **Remember Fubini!** $\displaystyle\int_0^4 -re^{r\cos\theta}\Big|_0^{\pi/2}\; dr = \int_0^4 -re^0 + re^r\; dr = -\frac{r^2}{2}\Big|_0^4 + (re^r - e^r)|_0^4 =$ $-8 + (4e^4 - e^4) - (0 - e^0) = -7 + 3e^4$ (4b.) This looks like oe of those hard "switch the limits of the triple integral" problem but since YOU get to pick which limits to switch the easiest choice is to go from $dz\; dy\; dx$ to $dz\; dx\; dy$ since the z limits won't have to change and this will be like a regular double integral that you apply Fubini to switch the order of integration. The integral becomes $\int_0^1\int_0^{y^2}\int_0^{1-y} dz\; dx\; dy = \int_0^1\int_0^{y^2} 1 - y\; dx\; dy = \int_0^1 y^2 - y^3\; dy = \displaystyle\frac{y^3}{3} - \frac{y^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

#5 Constrained Multivariable Optimization. The idea here was to find the maximum of a product of n numbers when the sum of this same n numbers must be constant. When you take the gradient of the sum constraint you will get $\vec{\nabla} g = (1, 1, 1, 1, 1, \ldots, 1)$ while when you take the gradient of the product you get $\frac{\partial f}{\partial x_k} = \Pi_{i=1, i\neq k}^{n} x_i$ plus the constrain $\displaystyle\sum_{i=1}^{n} x_i = c$ which is $n+1$ equations in $n+1$ unknowns $x_1, x_2, x_3, \ldots, x_n$ and λ. It turns out that $x_1 = x_2 = x_3 = \ldots = x_n$ OR $\lambda = 0$ solve the Lagrange Multiplier system of $n+1$ equations. $\lambda = 0$ is impossible (since one of the numbers would have to be zero). So you know that all the numbers have to equal c/n which is also their geometic, and arithmetic mean.

#EXTRA CREDIT Unconstrained Multivariable Optimization. This is one of the only straightforward Calculation problems. For $f(x, y) = x^4 + y^4 - 4xy + 1$, $\vec{\nabla} f(x, y) = (4x^3 - 4y, 4y^3 - 4x)$ which is equal to $\vec{0}$ at critical points. So $y = x^3$ and $x = y^3$. Thus $x = x^9$ so $x^9 - x = 0$ and $x(x^8 - 1) = x(x^4 + 1)(x^4 - 1) = x(x^4 + 1))(x^2 - 1)(x^2 + 1) = x(x^4 + 1)(x^2 + 1)(x - 1)(x + 1) = 0$ which means that $(0, 0)$, $(-1, -1)$ and $(1, 1)$ are the critical points. To classify the critical points you need $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = -4$. Evaluate $D = f_{xx}f_{yy} - f_{xy}^2$ at each of the three critical points and $D(0, 0) = -16 < 0$ (saddle point), $D(-1, -1) = D(1, 1) = 128 > 0$ (local min).