Linear Systems

Math 214 Spring 2008 | Fowler 309 MWF 9:30 am - 10:25 am

 | http://faculty.oxy.edu/ron/math/214/08/

Class 27

TITLE Projection Matrices and Orthogonal Diagonalization
CURRENT READING Poole 5.4

Summary
How to find the projection matrix for projection onto a subspace.

Homework Assignment
*HW #26: Poole, Section 5.4: 1,6,7,8,9,***11,12***,13,14,***22,23***. EXTRA CREDIT 25.*

1. Recalling Projection of a Vector onto Another Vector

For any vectors $\vec{u}$ and $\vec{v}$ where $\vec{u} \neq 0$ then **the projection of $\vec{v}$ onto $\vec{u}$** is the vector $\text{proj}_{\vec{u}}(\vec{v})$ defined by:

$$\text{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{\vec{u}^T\vec{v}}{\vec{u}^T\vec{u}}\right)\vec{u}$$

(Since $\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b} = \vec{b}^T\vec{a}$)

2. Projecting a Vector Onto A Subspace

What are the projections of $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ onto

(a) the z-axis, i.e. span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$?

(b) the xy-plane, i.e. span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$?

We can find P, a projection matrix, which finds the vector $\vec{p} = P\vec{b}$ which is the projection of $\vec{b}$ onto a vector space (e.g. the z axis, the xy-plane) spanned by a given basis $\mathcal{W}$ where $\vec{p} = \text{proj}_{\mathcal{W}}(\vec{b})$.

Consider $P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Do these matrices do what we want them to?

Note that the xy-plane is the column space of a matrix A, col(A) where A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, a matrix whose columns are a basis for the subspace consisting of the xy-plane.

Also note that the z-axis is the column space of the matrix $A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

In general, we are trying to find the projection of a vector $\vec{b}$ onto the column space of any $m \times n$ matrix (or the span of a given basis for a vector space).

3. Computing The Projection Matrix

Let's look more closely at that projection formula again:

$$\begin{aligned}
\text{proj}_{\vec{u}}(\vec{v}) &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u} \\
&= \left(\frac{\vec{u}^T\vec{v}}{\vec{u}^T\vec{u}}\right)\vec{u} \\
&= \frac{(\vec{u}^T\vec{v})\vec{u}}{\vec{u}^T\vec{u}} \\
&= \frac{\vec{u}(\vec{u}^T\vec{v})}{\vec{u}^T\vec{u}} \\
&= \frac{(\vec{u}\vec{u}^T)\vec{v}}{\vec{u}^T\vec{u}} \\
&= \left(\frac{\vec{u}\vec{u}^T}{\vec{u}^T\vec{u}}\right)\vec{v} \\
\text{proj}_{\vec{u}}(\vec{v}) &= P\vec{v}
\end{aligned}$$

So the projection matrix P for projecting a vector $\vec{v}$ onto a vector $\vec{u}$ is given by $P = \dfrac{\vec{u}\vec{u}^T}{\vec{u}^T\vec{u}}$.

Exercise

Use the above formula to find the projection matrix for the projection of the vector $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ onto the z-axis.

EXAMPLE

Let's change the formula to find the projection matrix onto the subspace spanned by span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, i.e. the xy-plane.

Projection Matrix Formula

So the projection matrix P for projecting a vector $\vec{v}$ onto a subspace equal to the col(A) is given by $P = A(A^TA)^{-1}A^T$.

Exercise

Find the Projection matrix P which computes the projection of a vector $\vec{v}$ onto the subspace $\mathcal{W}$ where $\mathcal{W}$ is the plane $x - y + 2z = 0$. Use your answer to obtain the projection of $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ onto $\mathcal{W}$. After doing that, we can also find the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $\mathcal{W}$ without doing much work. Why?

4. Orthogonal Diagonalization

DEFINITION

A square matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T A Q = D$ (or $A = QDQ^T$.)

EXAMPLE

Let's show the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ is orthogonally diagonalizable.

Theorem 5.18

If A is a real symmetric matrix, then the eigenvalues of A are real.

Theorem 5.19

If A is a symmetric matrix, then any two eigenvectors corresponding to two distinct eigenvalues of A are orthogonal.

Theorem 5.20

A real square matrix is symmetric IF and ONLY IF it is orthogonally diagonalizable. This result is known as **The Spectral Theorem.**

DEFINITION

The **spectrum** of a matrix is the set of all eigenvalues of the matrix. The **spectral decomposition** of a matrix A is the expression $A = \sum_{i=1}^{n} \lambda_i \vec{q}_i \vec{q}_i^T$ where $\vec{q}_1, \vec{q}_2, \vec{q}_3, \ldots, \vec{q}_n$ are the columns of the orthogonal matrix Q.

$$\begin{aligned}
A &= QDQ^T \\
&= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \ldots & \vec{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \\ \vdots \\ \vec{q}_n^T \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1\vec{q}_1 & \lambda_2\vec{q}_2 & \lambda_3\vec{q}_3 & \ldots & \lambda_n\vec{q}_n \end{bmatrix} \begin{bmatrix} \vec{q}_2^T \\ \vec{q}_2^T \\ \vec{q}_3^T \\ \vdots \\ \vec{q}_n^T \end{bmatrix} \\
&= \lambda_1\vec{q}_1\vec{q}_1^T + \lambda_2\vec{q}_2\vec{q}_2^T + \ldots + \lambda_n\vec{q}_n\vec{q}_n^T = \sum_{i=1}^{n} \lambda_i\vec{q}_i\vec{q}_i^T = \sum_{i=1}^{n} \lambda_i \left(\frac{\vec{q}_i\vec{q}_i^T}{\vec{q}_i^T\vec{q}_i} \right)
\end{aligned}$$

Does this expression look familiar? Each of the expressions $\vec{q}_i\vec{q}_i^T$ are rank 1 matrices which represent the projection matrix for projecting onto the 1-dimension space spanned by each $\vec{q}_i$. (Recall $\vec{q}_i^T\vec{q}_i = 1$). The above expression is thus sometimes known as the **projection form of the Spectral Theorem**.

Exercise

Let's find the spectral decomposition of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

CLICKER QUESTION 27.1

True or False If P is a projection matrix of the form $P = A(A^T A)^{-1} A^T$ then it is a symmetric matrix.

CLICKER QUESTION 27.2

True or False If P is a projection matrix of the form $P = A(A^T A)^{-1} A^T$ then it is an invertible matrix.

CLICKER QUESTION 27.3

True or False If A is a symmetric, invertible matrix, then $A^{-1} = A^T$.

CLICKER QUESTION 27.4

If A is an $n \times n$ real symmetric matrix, then which of the following is true?

1. Each eigenvalue of A is real.
2. If A is invertible, then its inverse is also symmetric.
3. If $Ax = 2x$ and $Ay = 3y$ then $x \cdot y = 0$.
4. If λ_1 and λ_2 are two different eigenvalues of A and W_1 and W_2 are the corresponding eigenspaces, then W_1 and W_2 are orthogonal sets.
5. All of the above are true.
6. More than one, but not all, of the above are true.