Linear Systems

Math 214 Spring 2008 Fowler 309 MWF 9:30 am - 10:25 am
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Class 26

TITLE Gram-Schmidt Orthogonalization Process and QR factorization
CURRENT READING Poole 5.3

Summary
There is a very cool algorithm for producing an orthogonal basis from a "regular" basis for a given subspace. The process is known as Gram-Schmidt orthogonalization. We also are introduced to *another* matrix factorization, $A = QR$.

Homework Assignment
*HW #25 Poole, Section 5.3 : 1,***2,3***,4,***6,11,13***, 17. EXTRA CREDIT 18.*

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1. Gram-Schmidt Orthogonalization

Suppose we start off with three linearly independent vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. First we will construct three orthogonal vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and then normalize these to produce three orthonormal vectors $\vec{q_1}$, $\vec{q_2}$ and $\vec{q_3}$ from our original linearly independent trio.

STEP 1. First choice, start with $\vec{a}$.
1. Let $\vec{A} = \vec{a}$.

STEP 2. Second choice, select in the direction of $\vec{b}$ with the projection in the direction of $\vec{a}$ **removed**. Then this vector should be orthogonal to $\vec{a}$.
2. Let $\vec{B} = \vec{b} - \left(\dfrac{\vec{A}\cdot\vec{b}}{\vec{A}\cdot\vec{A}}\right)\vec{A} = \vec{b} - \text{proj}_{\vec{A}}(\vec{b}) = \text{perp}_{\vec{A}}(\vec{b})$

STEP 3. Third choice, select in the direction of $\vec{c}$ with the projections of $\vec{c}$ in the direction of $\vec{a}$ and in the direction of $\vec{b}$ removed. So this third vector will be orthogonal to both of those!
3. Let $\vec{C} = \vec{c} - \left(\dfrac{\vec{A}\cdot\vec{c}}{\vec{A}\cdot\vec{A}}\right)\vec{A} - \left(\dfrac{\vec{B}\cdot\vec{c}}{\vec{B}\cdot\vec{B}}\right)\vec{B} = \vec{c} - \text{proj}_{\vec{A}}(\vec{c}) - \text{proj}_{\vec{B}}(\vec{c})$

STEP 4. Normalize A, B and C by dividing by their magnitudes to obtain $\vec{q_1} = \frac{\vec{A}}{||\vec{A}||}$, $\vec{q_2} = \frac{\vec{B}}{||\vec{B}||}$ and $\vec{q_3} = \frac{\vec{C}}{||\vec{C}||}$.
The vectors $\vec{q_1}$, $\vec{q_2}$ and $\vec{q_3}$ are **orthonormal**! Let's verify this.

EXAMPLE

Let's use Gram-Schmidt to convert the linearly independent vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\}$ to three orthonormal vectors.

Does it matter which vector you choose to start the orthogonalization process with?

2. A=QR Factorization

Gram-Schmidt Orthogonalization is equivalent to factoring a $m \times n$ matrix A into the product of a $m \times n$ matrix Q with orthonormal columns and R is an invertible $n \times n$ upper triangular matrix.

$$\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{q_1} & \vec{q_2} & \vec{q_3} \end{bmatrix} \begin{bmatrix} \vec{q_1}^T\vec{a} & \vec{q_1}^T\vec{b} & \vec{q_1}^T\vec{c} \\ & \vec{q_2}^T\vec{b} & \vec{q_2}^T\vec{c} \\ & & \vec{q_3}^T\vec{c} \end{bmatrix}$$

Exercise

Strang, page 230, #23. Find $\vec{q_1}$, $\vec{q_2}$, and $\vec{q_3}$ as combinations of the independent columns of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$ and write A as QR.

Why The QR Factorization Is Useful

If an $m \times n$ matrix A is able to be factored into $A = QR$ with the properties above (Q has orthonormal columns and R is an upper triangular invertible matrix) then this means that the computation of the solution of $A\vec{x} = \vec{b}$ can be simplified.

$$\begin{aligned}
A\vec{x} &= \vec{b} \quad \text{(The orginal linear system)} \\
QR\vec{x} &= \vec{b} \quad \text{(factor } A \text{ into the product of } Q \text{ and } R\text{)} \\
Q(R\vec{x}) &= \vec{b} \quad \text{(Let } \vec{y} = R\vec{x}\text{)} \\
Q\vec{y} &= \vec{b} \quad \text{(Now we have two simpler linear systems } y = Rx \text{ and } Qy = b \text{ instead of } Ax = b\text{)} \\
Q^TQ\vec{y} &= Q^T\vec{b} \quad \text{(But since } Q \text{ has orthonormal columns } Q^T = Q^{-1}\text{)} \\
\vec{y} &= Q^T\vec{b} \quad \text{(Since } Q^TQ = \mathcal{I}\text{)} \\
R\vec{x} &= Q^T\vec{b} \quad \text{(Recall that } \vec{y} = R\vec{x}\text{)} \\
\vec{x} &= R^{-1}Q^T\vec{b}
\end{aligned}$$

Finding the solution of the linear system will involve inverting an upper triangular matrix and taking a transpose. The QR factorization also occurs in the computation of eigenvalues numerically and the Least Squares approximation problem.

CLICKER QUESTION 26.1

Let A be an $m \times n$ matrix with linearly independent columns $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$.
Applying the Gram-Schmidt process to $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ will produce

1. an orthogonal basis for A.
2. an orthogonal basis for the column space of A.
3. an orthogonal basis for the row space of A.
4. an orthogonal basis for the null space of A.

CLICKER QUESTION 26.2

Let $v_1 = (2, -1, 0)$ and $v_2 = (1, 1, 1)$. The Gram-Schmidt process,
when applied to these vectors, produces $\{v_1', v_2'\}$ where

1. $v_1' = (2, -1, 0)$ and $v_2' = (-1, 2, 1)$.
2. $v_1' = (2, -1, 0)$ and $v_2' = (3/5, 6/5, 1)$.
3. $v_1' = (2, -1, 0)$ and $v_2' = (2/5, -1/5, 0)$.
4. $v_1' = (2, -1, 0)$ and $v_2' = (7/5, 6/5, 1)$.
5. $v_1' = (2, -1, 0)$ and $v_2' = (3/2, 3, 1)$.

CLICKER QUESTION 26.3

True or False

If $W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, then $S = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \right\}$ is an orthogonal basis for W.

CLICKER QUESTION 26.4

True or False The Gram-Schmidt Orthorgonlization process can be used to construct an orthonormal set of vectors from an arbitrary set of vectors.