Linear Systems

Math 214 Spring 2008 — Fowler 309 MWF 9:30 am - 10:25 am

 — http://faculty.oxy.edu/ron/math/214/08/

Class 25

TITLE Orthogonal Complements and Orthogonal Projections
CURRENT READING Poole 5.1

Summary
We will learn about an incredibly important feature of vectors and orthogonal vector spaces.

Homework Assignment
*HW#24 Poole, Section 5.2: 2,3,***4,5***,6,***7,12***,15,***16***,17,19,***20,21***. EXTRA CREDIT 29.*

DEFINITION

Two subspaces $\mathcal{V}$ and $\mathcal{W}$ are said to be **orthogonal** if every vector $\vec{v} \in \mathcal{V}$ is perpendicular to every vector $\vec{w} \in \mathcal{W}$. The **orthogonal complement** of a subspace $\mathcal{V}$ contains EVERY vector that is perpendicular to (vectors in) $\mathcal{V}$. This space is denoted $\mathcal{V}^\perp$. In other words, $\vec{v} \cdot \vec{w} = 0$ or $\vec{v}^T \vec{w} = 0$ for every $\vec{v}$ in $\mathcal{V}$ and $\vec{w}$ in $\mathcal{W}$.

$$\mathcal{W}^\perp = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \text{ in } \mathcal{W}\}$$

Example 1. **Q:** In $\mathbb{R}^3$, let $V =$ the z-axis. What is $V^\perp$? **A:** ________________________________

Q: In $\mathbb{R}^3$, what is the orthogonal complement of the xy-plane?
A: ________________________________

Q: In $\mathbb{R}^3$, are the xy-plane and the yz-plane orthogonal complements of each other?
A: No, there are vectors in one plane that are not perpendicular to vectors in the other plane. (Can you find one of each?)

Q: In $\mathbb{R}^4$ (with axes x_1, x_2, x_3, x_4), what is the orthogonal complement of the x_1x_2-plane?

A: ________________________________

We can summarize some of the properties of orthogonal complements.

Theorem 5.9

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$.
[**a.**] $\mathcal{W}^\perp$ is a subspace of $\mathbb{R}^n$
[**b.**] $(\mathcal{W}^\perp)^\perp = \mathcal{W}$
[**c.**] $(\mathcal{W}^\perp) \cap \mathcal{W} = \vec{0}$
[**d.**] If $\mathcal{W}$=span$(\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n)$ then $\vec{v}$ is in $\mathcal{W}^\perp$ only if $\vec{v} \cdot \vec{w}_i = 0$ for every $\vec{w}_i$ in $\mathcal{W}$ for $i = 1 \ldots n$

These features can be described using the associated subspaces of an $m \times n$ matrix A.

Theorem 5.10

Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the null space of A. The orthogonal complement of the column space of A is the null space of A^T (sometimes called the left null space). Mathematically, this can be written:

$$(\text{row}(A))^\perp = \text{null}(A) \text{ and } (\text{col}(A))^\perp = \text{null}(A^T)$$

These four subspaces are called the **fundamental subspaces of the matrix** A.

This page will have a reproduction of a diagram of the relationship of th four fundamental subpsaces from Gilbert Strang's *Linear Algebra* textbook.

$\boxed{\text{EXAMPLE}}$

Let's find bases for the four fundamental subspaces of the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$.

Suppose we know that $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\operatorname{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Write down the dimensions of each fundamental subspace and describe the subspace-orthogonal complement pairs.

$\boxed{\text{DEFINITION}}$

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n\}$ be an orthogonal basis for $\mathcal{W}$. For any vector $\vec{v}$ in $\mathbb{R}^n$, the orthogonal project of $\vec{v}$ onto $\mathcal{W}$ is defined as

$$\operatorname{proj}_{\mathcal{W}}(\vec{v}) = \sum_{j=1}^{n} \operatorname{proj}_{\vec{w}_j}(\vec{v}) = \sum_{j=1}^{n} \frac{\vec{v} \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \vec{w}_j$$

The **component of $\vec{v}$ orthogonal to $\mathcal{W}$** is the vector $\operatorname{perp}_{\mathcal{W}}(\vec{v}) = \vec{v} - \operatorname{proj}_{\mathcal{W}}(\vec{v})$

NOTE: this implies that $\vec{v} = \operatorname{perp}_{\mathcal{W}}(\vec{v}) + \operatorname{proj}_{\mathcal{W}}(\vec{v})$ (Draw a picture in $\mathbb{R}^2$!)

$\boxed{\textbf{Theorem 5.11}}$

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $\vec{v}$ be ANY vector in $\mathbb{R}^n$. THEN there exist unique vectors $\vec{w}$ in $\mathcal{W}$ and $\vec{w}^{\perp}$ in $\mathcal{W}^{\perp}$ such that $\vec{v} = \vec{w} + \vec{w}^{\perp}$. This theorem is known as the **Orthogonal Decomposition Theorem**. Note: a corollary of this theorem is that $(\mathcal{W}^{\perp})^{\perp} = \mathcal{W}$.

$\boxed{\text{EXAMPLE}}$

Consider the subspace $\mathcal{W}$, $x - y + 2z = 0$ with the vector $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. Show that the orthogonal decomposition of $\vec{v}$ is $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$

$\boxed{\textbf{Theorem 5.13}}$

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ then $\dim(\mathcal{W}) + \dim(\mathcal{W}^{\perp}) = n$.

A corollary of Theorem 5.13 becomes clear when one applies it to the associated subspaces of a $m \times n$ matrix A. This is known as **The Rank Theorem**.

$\dim(\text{row}(A)) + \dim(\text{null}(A)) = n$ and $\dim(\text{col}(A)) + \dim(\text{null}(A^T)) = m$

$\boxed{\textbf{The Rank Theorem}}$

If A is an $m \times n$ matrix, then $\text{rank}(A) + \text{nullity}(A) = n$ and $\text{rank}(A) + \text{nullity}(A^T) = m$.

(Recall, $\text{rank}(A) = \text{rank}(A^T)$)

CLICKER QUESTION 25.1

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. Which of the following vectors is orthogonal to the row space of A?

1. $(1, 1, -1)$
2. (1, 4, 2)
3. (0, 0, 5)
4. $(-1, 0, 1)$

CLICKER QUESTION 25.2

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. Which of the following vectors is orthogonal to the column space of A?

1. $(1, 1, -1)$
2. (1, 4, 2)
3. $(0, 1, -2)$
4. $(2, 0, 2)$

CLICKER QUESTION 25.3

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. Which of the following vectors is orthogonal to the nullspace of A?

1. $(1, 1, -1)$
2. (1, 4, 2)
3. $(0, 1, -2)$
4. $(2, 0, 2)$

CLICKER QUESTION 25.4

True or False Any set of nonzero orthogonal vectors must also be linearly independent.

CLICKER QUESTION 25.5

True or False The only orthonormal basis for $\mathbb{R}^2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

CLICKER QUESTION 25.6

Let Q be a square matrix with orthonormal columns. **True or False** $Q^{-1} = Q^T$.

CLICKER QUESTION 25.7

If $\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then the orthogonal projection of $\vec{b}$ onto $\vec{y}$ is

1. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$
3. $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$
4. $\begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix}$

CLICKER QUESTION 25.8

If $\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and l is the line $y = \frac{1}{2}x$, then the orthogonal projection of $\vec{b}$ onto l is

1. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$
3. $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$
4. $\begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix}$

CLICKER QUESTION 25.9

If l is the line $y = 3x$, $\vec{b} \in \mathbb{R}^2$, and z is the orthogonal projection of $\vec{b}$ on l, then which of the following are true?

1. $b - z$ is perpendicular to l.
2. $b - z$ is a point on l.
3. z is of the form $(c, 3c)$
4. Exactly two of the statements are true.
5. None of the above are true.