Class 25

TITLE Orthogonal Complements and Orthogonal Projections

CURRENT READING Poole 5.1

Summary
We will learn about an incredibly important feature of vectors and orthogonal vector spaces.

Homework Assignment
HW#24 Poole, Section 5.2: 2,3, 4,5, 6, 7,12,15,16,17,19,20,21. EXTRA CREDIT 29.

DEFINITION
Two subspaces \( V \) and \( W \) are said to be orthogonal if every vector \( \vec{v} \in V \) is perpendicular to every vector \( \vec{w} \in W \). The orthogonal complement of a subspace \( V \) contains EVERY vector that is perpendicular to (vectors in) \( V \). This space is denoted \( V^\perp \). In other words, \( \vec{v} \cdot \vec{w} = 0 \) or \( \vec{v}^T \vec{w} = 0 \) for every \( \vec{v} \) in \( V \) and \( \vec{w} \) in \( W \).

\[
W^\perp = \{ \vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \}
\]

Example 1. Q: In \( \mathbb{R}^3 \), let \( V = \) the z-axis. What is \( V^\perp \)? A: 

Q: In \( \mathbb{R}^3 \), what is the orthogonal complement of the xy-plane?
A: 

Q: In \( \mathbb{R}^3 \), are the xy-plane and the yz-plane orthogonal complements of each other?
A: No, there are vectors in one plane that are not perpendicular to vectors in the other plane. (Can you find one of each?)

Q: In \( \mathbb{R}^4 \) (with axes \( x_1, x_2, x_3, x_4 \)), what is the orthogonal complement of the \( x_1x_2 \)-plane?
A: 

We can summarize some of the properties of orthogonal complements.

**Theorem 5.9**
Let \( W \) be a subspace of \( \mathbb{R}^n \).

[a.] \( W^\perp \) is a subspace of \( \mathbb{R}^n \)

[b.] \( (W^\perp)^\perp = W \)

[c.] \( (W^\perp) \cap W = \vec{0} \)

[d.] If \( W = \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n) \) then \( \vec{v} \) is in \( W^\perp \) only if \( \vec{v} \cdot \vec{w}_i = 0 \) for every \( \vec{w}_i \) in \( W \) for \( i = 1 \ldots n \)

These features can be described using the associated subspaces of an \( m \times n \) matrix \( A \).

**Theorem 5.10**
Let \( A \) be an \( m \times n \) matrix. Then the orthogonal complement of the row space of \( A \) is the null space of \( A \). The orthogonal complement of the column space of \( A \) is the null space of \( A^T \) (sometimes called the left null space). Mathematically, this can be written:

\[
(\text{row}(A))^\perp = \text{null}(A) \quad \text{and} \quad (\text{col}(A))^\perp = \text{null}(A^T)
\]

These four subspaces are called the fundamental subspaces of the matrix \( A \).
This page will have a reproduction of a diagram of the relationship of the four fundamental subspaces from Gilbert Strang's *Linear Algebra* textbook.
EXAMPLE

Let’s find bases for the four fundamental subspaces of the matrix \( A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \).

Suppose we know that \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \) and \( \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \). Write down the dimensions of each fundamental subspace and describe the subspace-orthogonal complement pairs.

DEFINITION

Let \( \mathcal{W} \) be a subspace of \( \mathbb{R}^n \) and let \( \{ \vec{w}_1, \vec{w}_2, \vec{w}_3, \ldots, \vec{w}_n \} \) be an orthogonal basis for \( \mathcal{W} \). For any vector \( \vec{v} \) in \( \mathbb{R}^n \), the orthogonal project of \( \vec{v} \) onto \( \mathcal{W} \) is defined as

\[
\text{proj}_{\mathcal{W}}(\vec{v}) = \sum_{j=1}^{n} \text{proj}_{\vec{w}_j}(\vec{v}) = \sum_{j=1}^{n} \frac{\vec{v} \cdot \vec{w}_j}{\vec{w}_j \cdot \vec{w}_j} \vec{w}_j
\]

The component of \( \vec{v} \) orthogonal to \( \mathcal{W} \) is the vector \( \text{perp}_{\mathcal{W}}(\vec{v}) = \vec{v} - \text{proj}_{\mathcal{W}}(\vec{v}) \)

NOTE: this implies that \( \vec{v} = \text{perp}_{\mathcal{W}}(\vec{v}) + \text{proj}_{\mathcal{W}}(\vec{v}) \) (Draw a picture in \( \mathbb{R}^2 \! \)!)
Theorem 5.11
Let \( W \) be a subspace of \( \mathbb{R}^n \) and let \( \vec{v} \) be ANY vector in \( \mathbb{R}^n \). THEN there exist unique vectors \( \vec{w} \) in \( W \) and \( \vec{w}^\perp \) in \( W^\perp \) such that \( \vec{v} = \vec{w} + \vec{w}^\perp \). This theorem is known as the **Orthogonal Decomposition Theorem**. Note: a corollary of this theorem is that \( (W^\perp)^\perp = W \).

**EXAMPLE**
Consider the subspace \( W, x - y + 2z = 0 \) with the vector \( \vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \). Show that the orthogonal decomposition of \( \vec{v} \) is
\[
\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}
\]

Theorem 5.13
Let \( W \) be a subspace of \( \mathbb{R}^n \) then \( \dim(W) + \dim(W^\perp) = n \).

A corollary of Theorem 5.13 becomes clear when one applies it to the associated subspaces of a \( m \times n \) matrix \( A \). This is known as the **Rank Theorem**.
\[
\dim(\text{row}(A)) + \dim(\text{null}(A)) = n \text{ and } \dim(\text{col}(A)) + \dim(\text{null}(A^T)) = m
\]

**The Rank Theorem**
If \( A \) is an \( m \times n \) matrix, then \( \text{rank}(A) + \text{nullity}(A) = n \) and \( \text{rank}(A) + \text{nullity}(A^T) = m \).

(Recall, \( \text{rank}(A) = \text{rank}(A^T) \))
**CLICKER QUESTION 25.1**

Let \( A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \). Which of the following vectors is orthogonal to the row space of \( A \)?

1. (1, 1, −1)
2. (1, 4, 2)
3. (0, 0, 5)
4. (−1, 0, 1)

**CLICKER QUESTION 25.2**

Let \( A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \). Which of the following vectors is orthogonal to the column space of \( A \)?

1. (1, 1, −1)
2. (1, 4, 2)
3. (0, 1, −2)
4. (2, 0, 2)

**CLICKER QUESTION 25.3**

Let \( A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \). Which of the following vectors is orthogonal to the nullspace of \( A \)?

1. (1, 1, −1)
2. (1, 4, 2)
3. (0, 1, −2)
4. (2, 0, 2)

**CLICKER QUESTION 25.4**

**True or False** Any set of nonzero orthogonal vectors must also be linearly independent.

**CLICKER QUESTION 25.5**

**True or False** The only orthonormal basis for \( \mathbb{R}^2 \) is \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \).

**CLICKER QUESTION 25.6**

Let \( Q \) be a square matrix with orthonormal columns. **True or False** \( Q^{-1} = Q^T \).
CLICKER QUESTION 25.7

If $\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then the orthogonal projection of $\vec{b}$ onto $\vec{y}$ is

1. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$
3. $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$
4. $\begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix}$

CLICKER QUESTION 25.8

If $\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $l$ is the line $y = \frac{1}{2}x$, then the orthogonal projection of $\vec{b}$ onto $l$ is

1. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$
3. $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$
4. $\begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix}$

CLICKER QUESTION 25.9

If $l$ is the line $y = 3x$, $\vec{b} \in \mathbb{R}^2$, and $z$ is the orthogonal projection of $\vec{b}$ on $l$, then which of the following are true?

1. $b - z$ is perpendicular to $l$.
2. $b - z$ is a point on $l$.
3. $z$ is of the form $(c, 3c)$
4. Exactly two of the statements are true.
5. None of the above are true.