## *Class 24*

**TITLE** Orthogonality and Projections Revisited
**CURRENT READING** Poole 5.1


**Summary**
We shall return to the investigation of projections and orthogonality, this time with more increased generality.


*Homework Assignment*
*HW #23 Poole, Section 5.1: 3,4,5,6,8,***9,12***,13,16,***17,30,31**. *EXTRA CREDIT 28, 33.*

### 1. Orthogonal Bases

DEFINITION

An **orthogonal basis** of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is a basis of $\mathcal{W}$ that is an **orthogonal set** of vectors. An orthogonal set of vectors is a collection of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ where *every* pair of distinct vectors is orthogonal to each other, i.e. $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$.

**Theorem 5.1**

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ then those vectors are linearly independent.

EXAMPLE

Show that $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^3$.

$\boxed{\textbf{Theorem 5.2}}$
Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in $\mathcal{W}$. THEN the unique scalars $c_1, c_2, c_3, \ldots, c_n$ (also known as coordinates) where $\vec{w} = \sum_{i=1}^{n} c_i\vec{v}_i$ are given by

$$c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

$\boxed{\text{EXAMPLE}}$
Let's show how this formula for the coordinates is derived. (Doesn't it look familiar??)

$\boxed{\textbf{Exercise}}$
Given the orthogonal basis $\beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and the vector $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ find the coordinates of $\vec{w}$ with respect to $\beta$, i.e. $[\vec{w}]_\beta$.

$\boxed{\text{DEFINITION}}$
An **orthonormal basis** of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is a basis of $\mathcal{W}$ that consists of an **orthonormal set** of vectors. An orthonormal set of vectors is a collection of orthogonal unit vectors $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \ldots, \vec{q}_k\}$ where $\vec{v}_i \cdot \vec{v}_j = \delta_{i,j}$. The symbol $\delta_{i,j}$ is known as the Kronecker delta function and has the property that $\delta_{i,j} = 0$ when $i \neq j$ and $\delta_{i,j} = 1$ when $i = j$.

$\boxed{\textbf{Exercise}}$
Form an orthonormal basis for $\mathbb{R}^3$ from the orthogonal basis $\beta$ given in the previous **Exercise**.

## 2. Orthogonal Matrices

DEFINITION

A $n \times n$ matrix $Q$ is said to be an **orthogonal matrix** if the columns (and rows) of the matrix form an orthonormal set.

**Theorem 5.4**

The columns of an $m \times n$ matrix $Q$ form an orthonormal set if and only if $Q^T Q = I_n$.

**Theorem 5.5**

A square matrix $Q$ is orthogonal if and only if $Q^{-1} = Q^T$.

**Theorem 5.8**

Let $Q$ be an orthogonal matrix. THEN

(a) $Q^{-1}$ is orthogonal.

(b) $\det(Q) = \pm 1$.

(c) If $\lambda$ is an eigenvalue of $Q$, then $|\lambda| = 1$.

(d) If $Q_1$ and $Q_2$ are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$.

EXAMPLE

Let's form a square orthogonal matrix from the orthonormal basis found in the previous exercise and illustrate some of the results from Theorem 5.4, 5.5 and 5.8.

$\boxed{\textbf{CLICKER QUESTION 24.1}}$

Which of the following sets of vectors is **not** an orthogonal set?

1. $(1, 1, 1), (1, 0, -1)$
2. $(2, 3), (-6, 4)$
3. $(3, 0, 0, 2), (0, 1, 0, 1)$
4. $(0, 2, 0), (-1, 0, 3)$
5. $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$

$\boxed{\textbf{CLICKER QUESTION 24.2}}$

**True or False** If two vectors are linearly independent, they must be orthogonal.

1. True
2. False.

$\boxed{\textbf{CLICKER QUESTION 24.3}}$

**True or False** Any orthogonal set of nonzero vectors that spans a vector space must be a basis for that space.

1. True
2. False.

$\boxed{\textbf{CLICKER QUESTION 24.4}}$

Which of the following sets of vectors is an orthonormal set?

1. $(1, 1, 1), (1, 0, -1)$
2. $(2, 3), (-6, 4)$
3. $(0, 2, 0), (-1, 0, 3)$
4. $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$

$\boxed{\textbf{CLICKER QUESTION 24.5}}$

Let $A$ be a square matrix whose column vectors are not zero and mutually orthogonal.
Which of the following are true?

1. The dot product of any two different column vectors is zero.
2. The set of column vectors is linearly independent.
3. $\det(A) \neq 0$.
4. For any $b$, there is a unique solution to $Ax = b$.
5. All of the above.