Linear Systems

Class 23

TITLE Diagonalization and Similarity
CURRENT READING Poole 4.4

Summary
One application of computing eigenvalues and eigenvectors leads to an important matrix factorization and characteristic of a matrix known as "diagonalizability."

Homework Assignment
HW#22: Poole, Section 4.4: **2**,*5*,*6*, **9**, **10**, **16**,**18**,*21*,*22*,*24*,*25*. *EXTRA CREDIT 23.*

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1. Factoring A $=S\Lambda S^{-1}$

S is a matrix whose columns consist of the eigenvectors of A.
Λ is a diagonal matrix with the eigenvalues of A along the diagonal.
The factorization is only possible if the $n \times n$ (square) matrix A has exactly n *linearly independent* eigenvectors. In other words, none of the eigenvectors can be a linear combination of the other eigenvectors (other wise S^{-1} would not exist).
Let's show that $A = S\Lambda S^{-1}$ and $AS = S\Lambda$ **and** $\Lambda = S^{-1}AS$. This last form is the most important, because it means that we can produce a diagonal matrix Λ from a given square matrix A by pre- and post- multiplying it by the special matrix S. This process is called **diagonal decomposition**.

Proof
If $\vec{x_1}, \vec{x_2}, \vec{x_3}, \ldots, \vec{x_n}$ are n linearly independent eigenvectors of A which make up the columns of a special matrix S then

$$\begin{aligned} AS &= A\begin{bmatrix} \vec{x_1} & \vec{x_2} & \vec{x_3} & \ldots & \vec{x_n} \end{bmatrix} = \begin{bmatrix} A\vec{x_1} & A\vec{x_2} & A\vec{x_3} & \ldots & A\vec{x_n} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\vec{x_1} & \lambda_2\vec{x_2} & \lambda_3\vec{x_3} & \ldots & \lambda_n\vec{x_n} \end{bmatrix} = \begin{bmatrix} \vec{x_1} & \vec{x_2} & \vec{x_3} & \ldots & \vec{x_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} = S\Lambda \end{aligned}$$

The diagonalization matrix factorization $A = S\Lambda S^{-1}$ is a special case of **similar matrices**.

DEFINITION
A is said to be **similar** to B if there exists an invertible $n \times n$ matrix P so that $B = P^{-1}AP$ (and thus $PB = AP$ or $AP = PB$). If A is similar to B we say that $A \sim B$.

The process of diagonalization is finding a diagonal matrix which is similar to the given $n \times n$ matrix A.

EXAMPLE Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ with eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$.

2. Similar Matrices

Theorem 4.21

Let A, B and C be $n \times n$ matrices. Recall that A is similar to B, i.e. $A \sim B$ if there exists an invertible $n \times n$ matrix P such that $P^{-1}AP = B$.
(i) $A \sim A$ (**Reflexivity**)
(ii) If $A \sim B$, then $B \sim A$ (**Symmetry**)
(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$ (**Transitivity**)
You'll see more about these words (reflexive, symmetric and transitive) in **Math 210**! If a relation $\sim$ satisfies these properties it is known as an **equivalence relation**.

Exercise Can you prove each of the results in Theorem 4.21? You should be able to!

Theorem 4.22

Let A and B be two similar $n \times n$ matrices. THEN
(a) $\det(A) = \det(B)$
(b) A is invertible if and only if B is invertible.
(c) A and B have the same rank.
(d) A and B have the same characteristic polynomial.
(e) A and B have the same eigenvalues.

EXAMPLE Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Let's show that A and B have the same characteristic polynomial, the same eigenvalues, are both invertible, have rank 2 and the same determinant.

Q: Are these two matrices A and B similar to each other?
A: No! Does this mean that Theorem 4.22 is a vicious lie? Explain the apparent contradiction.

3. Matrix Exponentiation

One useful result of diagonal decomposition is that it allows us to compute values of A^n very easily. It is very easy to exponentiate a diagonal matrix.

$A^{10} = (S\Lambda S^{-1})^{10} = (S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots (S\Lambda S^{-1})$

Can we simplify this expression? YES!

$A^{10} = S\Lambda^{10}S^{-1}$

EXAMPLE

Compute $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^{10}$

4. More on Diagonalization

Theorem 4.25

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Theorem 4.26

The geometric multiplicity (the dimension of the eigenspace) of each eigenvalue is always less than or equal to the algebraic multiplicity (the multiplicity of the eigenvalue as a root of the characteristic polynomial).

Theorem 4.27

Let A be an $n \times n$ matrix with k distinct eigenvalues. The following statements are equivalent:
(a) A is diagonalizable.
(b) The union β of the bases of the eigenspaces of A contains n vectors.
(c) The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

GroupWork

Consider $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. Are either of these matrices diagonalizable? Explain your answer!

One application of matrix diagonalization is the computation of the matrix exponential, e^A. Similar to the definition of $A^n = S\Lambda^n S^{-1}$, if A is diagonalizable, then it has n linearly independent eigenvectors to make up the columns of S and thus

$$e^A = S \begin{bmatrix} e^{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_n} \end{bmatrix} S^{-1}$$

EXAMPLE

Let's compute e^A, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

5. Symmetric matrices are always diagonalizable

Consider the matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & d \end{bmatrix}$. Show that it has eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ d-1 \end{bmatrix}$ with eigenvalues $-1, 1, d$ respectively.

When $d \to 1$ the third eigenvector (and eigenvalue) collapses to be the same as the second, so that the S matrix for A will be singular and thus A will not be diagonalizable.

However, now consider the symmetric matrix $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & d \end{bmatrix}$. Show that it has eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with eigenvalues $-1, 1, d$ respectively.

As $d \to 1$ the second eigenvalue repeats, but the eigenvectors are unaffected. Note again: The eigenvectors are perpendicular (i.e. orthogonal) to each other so the matrix B can be diagonalized. The S matrix of eigenvectors will be non-singular and thus S^{-1} will exist. **Do it!**

$\boxed{\textbf{CLICKER QUESTION 23.1}}$

What are the eigenvalues of $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$?

1. 2 and 3
2. 0 and 2
3. 0 and 3
4. 5 and 6

$\boxed{\textbf{CLICKER QUESTION 23.2}}$

If $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, what is D^5?

1. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 10 & 0 \\ 0 & 15 \end{bmatrix}$
3. $\begin{bmatrix} 2^5 & 0 \\ 0 & 3^5 \end{bmatrix}$
4. Too hard to compute by hand.

$\boxed{\textbf{CLICKER QUESTION 23.3}}$

Which of the following statements are true?

1. An $n \times n$ matrix with n linearly independent eigenvectors is diagonalizable.
2. Any diagonalizable $n \times n$ matrix has n linearly independent eigenvectors.
3. Both are true.
4. Neither is true.

$\boxed{\textbf{CLICKER QUESTION 23.4}}$

Which of the following statements are true?

1. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.
2. Any diagonalizable $n \times n$ matrix has n distinct eigenvalues.
3. Both are true.
4. Neither is true.

CLICKER QUESTION 23.5

Which of the following statements are true?

1. If A is a diagonalizable matrix, then A does not have any zero eigenvalues.
2. If A does not have any zero eigenvalues, then A is diagonalizable.
3. Both are true.
4. Neither is true.

CLICKER QUESTION 23.6

True or False All invertible matrices are diagonalizable.

CLICKER QUESTION 23.7

True or False All diagonalizable matrices are invertible.

CLICKER QUESTION 23.8

Suppose $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is similar to $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ then which of the following statements are true?

1. $\det(A) = \det(D)$
2. A and D have the same eigenvalues.
3. There exists a matrix such that $PA = DP$.
4. All of these statements are true.
5. Some of these statements are true.
6. None of these statements are true.