Linear Systems

Math 214 Spring 2008 Fowler 309 MWF 9:30 am - 10:25 am

http://faculty.oxy.edu/ron/math/214/08/

Class 22: **Friday March 28**

TITLE More Eigenvalues and Eigenvectors
CURRENT READING Poole 4.3

Summary
Let's explore the wonderful world of *eigenvectors*, *eigenvalues* and *eigenspaces* of a square $n \times n$ matrix.

Homework Assignment
*HW#21 Poole, Section 4.3: 4,***5,10***,15,16,***17,18***,20,21,***23,33**. *EXTRA CREDIT 34,36,38.*

DEFINITION
The eigenvalues of a square $n \times n$ matrix A satisfy the **characteristic polynomial** of the matrix A, given by $\det(A - \lambda I) = 0$.

EXAMPLE

Find the eigenvalues and corresponding eigenspaces of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$

DEFINITION
The **algebraic multiplicity** of an eigenvalue is the multiplicity of this eigenvalue as a root of the characteristic polynomial. The **geometric muliplicity** of an eigenvalue λ is the *dimension* of the corresponding eigenspace E_λ, i.e. the number of vectors in a basis for the eigenspace.

Exercise

Write down the algebraic and geometric multiplicity of the eigenvalues of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$.

Theorem 4.15

The eigenvalues of a triangular matrix (lower triangular, upper triangular or diagonal) are simply the entries along its main diagonal.

Theorem 4.16

Let A be a square matrix with eigenvalue λ and eigenvector $\vec{x}$

(i) For any integer n, λ^n is an eigenvalue of A^n with correspnding eigenvector $\vec{x}$

(ii) If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$

Theorem 4.18

A square matrix A is invertible if and only if 0 is NOT an eigenvalue of A.

EXAMPLE

Poole, page 296, #19. (a) Show that for any square matrix A, A^T and A have the same characteristic polynomial and thus the same eigenvalues.

(b) Give an example of a 2x2 matrix A for which A^T and A have different eigenspaces.

Exercise

Show that the eigenvalues $A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$ are 5 and -2 and $E_{-2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right)$ and $E_5 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Find the eigenvalues of $3A$, A^{-1}, A^2 and $A + I$

Linear Independence of Eigenvectors

Theorem 4.19

Suppose the $n{\times}n$ matrix A has m eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. IF $\vec{x}$ is a vector in $\mathbb{R}^n$ that can be written as a linear combination of these vectors, THEN

$$A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + c_3\lambda_3^k\vec{v}_3 + \ldots c_m\lambda_m^k\vec{v}_m$$

EXAMPLE

Let's use this result to show that $\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}^6 \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 46747 \\ 47195 \end{bmatrix}$

Theorem 4.20

Let A be an $n \times n$ matrix with m distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$. Then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_m$ are linearly independent.

Properties of the Eigenvalues of a $n \times n$ Matrix

The **Product** of the eigenvalues equals the **determinant** of the $n \times n$ matrix.

$$\lambda_1\lambda_2\lambda_3\ldots\lambda_n = |A|$$

The **Sum** of the eigenvalues equals the **trace** of the $n \times n$ matrix (the sum of the diagonal entries)

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \sum_{i=1}^{n} A_{ii}$$

CLICKER QUESTION 22.1

The matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ has an eigenvalue 3 with associated eigenvector $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Which of the following statements is true?

1. $Ax = 3x$
2. $Ay = 3y$
3. For any scalars c and d, $A(cx + dy) = 3(cx + dy)$
4. All of the above are true.
5. Only (a) and (b) are true.

CLICKER QUESTION 22.2

If w is an eigenvector of A, how does the vector Aw compare geometrically to the vector w?

1. Aw is a rotation of w.
2. Aw is a reflection of w in the x-axis.
3. Aw is a reflection of w in the y-axis.
4. Aw is parallel to w but may have a different length.

CLICKER QUESTION 22.3

If a vector x is in the eigenspace of A corresponding to λ, then x is

1. in the nullspace of the matrix A.
2. in the nullspace of the matrix $A - \lambda I$.
3. not the zero vector.
4. More than one of the above correctly completes the sentence.

CLICKER QUESTION 22.4

Which of the following statements is correct?

1. The set of eigenvectors of a matrix A forms the eigenspace of A.
2. The set of eigenvectors of a matrix A spans the eigenspace of A.
3. Since any multiple of an eigenvector is also an eigenvector, the eigenspace always has infinite dimension.
4. More than one of the above statements are correct.
5. None of the above statements are correct.