Linear Systems

Math 214 Spring 2008 — Fowler 309 MWF 9:30 am - 10:25 am

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Class 18: **Friday March 23**

TITLE Introduction to Linear Transformations
CURRENT READING Poole 3.6

Summary
We'll introduce the concept of linear transformation. This is a very cool, visually interesting application of linear algebra.

Homework Assignment
*HW 17: Poole, Section 3.6: 1,***2***,3,***6***,***7***,15,17,***19***. EXTRA CREDIT 28.* DUE MON MAR 24.

Warm-Up
What's a function (broadly defined)? Can you think of function which has a $n \times n$ matrix as its input and a number as an output? What about a function which has a 5×1 vector as input and a 3×1 vector as output?

DEFINITION
A **transformation** or **function** or **mapping** from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{v} \in \mathbb{R}^n$ a unique vector $T(\vec{v}) \in \mathbb{R}^m$. The **domain** of T is $\mathbb{R}^n$ and the **co-domain** is $\mathbb{R}^m$. This is denoted $T : \mathbb{R}^n \to \mathbb{R}^m$. Given a vector $\vec{v}$ in the domain of T, the vector $T(\vec{v})$ is called the **image** of $\vec{v}$ under the action of T. The set of all possible images $T(\vec{v})$ is called the **range** of T.

DEFINITION
A transformation (or mapping or function) $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars c.

EXAMPLE
Given $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$. If we multiply A by an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$ we can define a transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

What is the image of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$? What is the pre-image of $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$? What is the range of T_A? Is T_A a linear transformation?

Theorem 3.30

Given a $m \times n$ matrix, the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\vec{x}) = A\vec{x}$ (for all $\vec{x}$ in $\mathbb{R}^n$) is a linear transformation.

Theorem 3.31

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as a matrix transformation T_A where A is the $m \times n$ matrix whose columns are given by the images of the standard basis vectors in $\mathbb{R}^n$ under the action of T, i.e.

$$A = \left[\begin{array}{ccccc} T(\hat{e}_1) & T(\hat{e}_2) & T(\hat{e}_3) & \ldots & T(\hat{e}_n) \end{array} \right]$$

This matrix is called the **standard matrix of the linear transformation** T and can be denoted $[T]$.

Exercise

Find the standard matrix for the linear transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ which maps each point to its reflection in the x-axis.

Find the standard matrix for the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates each point 90^o counterclockwise about the origin.

EXAMPLE

Let's find the standard matrix for the linear transformation R_θ which rotates a point θ degrees counterclockwise about the origin.

Composition of Linear Transformations

Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^p$ are linear transformations. Then the **composition** of the two transformations is denoted $S \circ T$. This means that a vector in the domain of T is mapped into the co-domain of S. The interpretation of the composition is a vector $\vec{v}$ acted on by T which is then acted on by S, i.e. $(S \circ T)(\vec{v}) = S(T(\vec{v}))$

Theorem 3.32

The standard matrix of a composition of linear transformations is equal to the product of the standard matrices of each transformation. $[S \circ T] = [S][T]$

EXAMPLE

Let's find the standard matrix of the linear transformation that first rotates a point 90^o about the origin and then reflects the result in the x-axis.

DEFINITION

The **identity transformation** is the transformation $I : \mathbb{R}^n \to \mathbb{R}^n$ which leaves every vector unchanged. If S and T are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ so that $S \circ T = I$ and $T \circ S = I$ then S and T are **inverse transformations** of each other. Both S and T are said to be invertible linear transformations.

Theorem 3.33

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then the standard matrix $[T]$ is an invertible matrix, and $[T^{-1}] = [T]^{-1}$. In other words, the standard matrix for the inverse linear transformation of T is the inverse of the standard matrix for the linear transformation T.

Exercise

Show that the linear transformation which maps a point θ degrees counterclockwise in $\mathbb{R}^2$ about the origin is **the inverse** of the linear transformation which maps a point θ degrees *clockwise*.

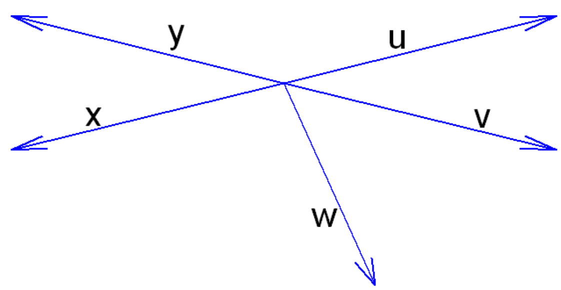
CLICKER QUESTION 18.1

Define $T(\vec{v}) = A\vec{v}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $T(\vec{v})$

1. reflects $\vec{v}$ about the x_2-axis.
2. reflects $\vec{v}$ about the x_1-axis.
3. rotates $\vec{v}$ clockwise $\pi/2$ radians about the origin.
4. rotates $\vec{v}$ counterclockwise $\pi/2$ radians about the origin.
5. None of the above

CLICKER QUESTION 18.2

Define $T(\vec{u}) = A\vec{u}$, where $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Using the vectors plotted below, this means that



1. $T(u) = v$.
2. $T(u) = w$.
3. $T(u) = x$.
4. $T(u) = y$.
5. None of the above

CLICKER QUESTION 18.3

If the linear transformation $T(v) = Av$ rotates the vectors $v_1 = (-1, 0)$ and $v_2 = (0, 1)$ clockwise $\pi/2$ radians, the resulting vectors are

1. $T(v_1) = \left(-\sqrt{2}/2, \sqrt{2}/2\right)$ and $T(v_2) = \left(\sqrt{2}/2, \sqrt{2}/2\right)$
2. $T(v_1) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right)$ and $T(v_2) = \left(-\sqrt{2}/2, \sqrt{2}/2\right)$
3. $T(v_1) = (0, -1)$ and $T(v_2) = (-1, 0)$
4. $T(v_1) = (0, 1)$ and $T(v_2) = (1, 0)$
5. None of the above

CLICKER QUESTION 18.4

The linear transformation $T(x, y) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, can be written as

1. $T(x, y) = (x, y)$
2. $T(x, y) = (y, x)$
3. $T(x, y) = (-x, y)$
4. $T(x, y) = (-y, x)$
5. None of the above

CLICKER QUESTION 18.5

The linear transformation $T(x, y) = (x + 2y, x - 2y)$, can be written as

1. $T(x, y) = \begin{bmatrix} x & 2y \\ x & -2y \end{bmatrix}$
2. $T(x, y) = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$
3. $T(x, y) = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$
4. It can't be written in matrix form

CLICKER QUESTION 18.6

Which of the following is not a linear transformation?

1. $T(x, y) = (x, y + 1)$
2. $T(x, y) = (x - 2y, x)$
3. $T(x, y) = (4y, x - 2y)$
4. $T(x, y) = (x, 0)$
5. All are linear transformations
6. None are linear transformations

CLICKER QUESTION 18.7

Is the transformation $T(f) = f'$ linear?

1. No, it is not linear because it does not satisfy the scalar multiplication property.
2. No, it is not linear because it does not satisfy the vector addition property.
3. No, it is not linear for a reason not listed here.
4. Yes, it is linear.