Linear Systems

Math 214 Spring 2008 | Fowler 309 MWF 9:30 am - 10:25 am

 | http://faculty.oxy.edu/ron/math/214/08/

Class 17: **Friday March 7**

TITLE Subspaces Associated With Matrices; Dimension and Basis
CURRENT READING Poole 3.5 and 6.1

Summary
Let's continue discussing vector spaces associated with matrices and formally define the concept of dimension.

Homework Assignment
HW16: Poole, Section 3.5: **17, 18**, *21, 24, 39,* **40, 41**, *42. EXTRA CREDIT 44, 50.*

Recall: Let A be an $m \times n$ matrix. The **row space** of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A and is denoted row(A). The **column space** of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A and is denoted col(A).

Warm-Up

Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 7 \end{bmatrix}$. Find col$(A)$ and row(A).

DEFINITION

The **null space** of a $m \times n$ matrix is the subspace of $\mathbb{R}^n$ consisting of all solutions of the homogeneous linear system $A\vec{x} = \vec{0}$. It is denoted by null(A).

Theorem 3.21

The set N of all solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ where A is a $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

Exercise

Prove Theorem 3.21 (that the nullspace of a matrix A is a subspace of $\mathbb{R}^n$).

DEFINITION

The **basis** of a subspace $\mathcal{S}$ of $\mathbb{R}^n$ is a set of vectors in $\mathcal{S}$ which is **linear independent** and **spans** $\mathcal{S}$. The plural of basis is bases.

Q: Are bases for a subspace unique? **A:** Heck, no! (Why not?)

GROUPWORK

Write down three examples of bases for $\mathbb{R}^2$.

Theorem 3.23

The number of vectors found in a basis for a subspace $\mathcal{S}$ of $\mathbb{R}^n$ is the same. Any two bases for $\mathcal{S}$ have the same number of vectors.

DEFINITION

The number of vectors in a basis for a subspace $\mathcal{S}$ of $\mathbb{R}^n$ is known as the **dimension** of $\mathcal{S}$ and is denoted $\dim(S)$. This result is known as the **Basis Thorem**.

EXAMPLE

Given $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Let's write down bases for $\text{col}(A)$, $\text{row}(A)$ and $\text{null}(A)$.

Theorem 3.20

If B is a matrix formed from applying elementary row operations to A and thus B is row equivalent to A, then $\text{row}(A) = \text{row}(B)$.

Elementary row operations **do not affect** the row space of a matrix, but they **do change** the column space of a matrix. Given $R = \text{rref}(A)$. $\text{row}(A) = \text{row}(R)$ but $\text{col}(A) \neq \text{col}(R)$.

Understanding Dimension

Previously we have conceived as the dimension of an object as the number of unknown parameters it takes to describe the object in the vector form of the equation. Now, that we know about subspaces and bases of subspaces we can be more precise.

GROUPWORK

Consider the following sets, find the dimension of each. (Recall matrix A from the previous EXAMPLE.)

1. row(A) **2.** col(A) **3.** null(A)

4. $\mathbb{R}^n$ **5.** $\{\vec{0}\}$ **6.** The set of all 2x3 matrices.

7. A line through the origin. **8.** A plane through the origin. **9.** A plane not through the origin.

Theorem 3.24

The row and column spaces of a matrix A have the same dimension. The **rank** of a matrix A is the dimension of its row and column spaces and is denoted rank(A). This is the same rank previously defined as the number of non-zero rows in the reduced row echelon form of A.

EXAMPLE

The rank of a matrix is the same as the rank of the transpose of the matrix, i.e., $\text{rank}(A) = \text{rank}(A^T)$. Can we prove this?

DEFINITION

The dimension of the null space of a matrix A is called the **nullity** of a matrix and is denoted nullity(A).

Theorem 3.26

For any $m \times n$ matrix A, $\text{rank}(A) + \text{nullity}(A) = n$. In other words, the dimension of the column space plus the dimension of the null space equals the number of column of the matrix. This result is known as **The Rank Theorem**.

Theorem 3.27

The Fundamental Theorem of Invertible Matrices (Version 2). Let A be a $n \times n$ matrix. Each of the following statements is equivalent:
(a) A is invertible.
(b) $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^n$.
(c) $A\vec{x} = \vec{0}$ has only the trivial solution.
(d) The reduced row echelon form of A, rref(A), is I_n.
(e) A is a product of elementary matrices.
(f) $\text{rank}(A) = n$.
(g) $\text{nullity}(A) = 0$.
(h) The column vectors of A are linearly independent.
(i) The column vectors of A span $\mathbb{R}^n$.
(j) The column vectors of A form a basis for $\mathbb{R}^n$.
(k) The row vectors of A are linearly independent.
(l) The row vectors of A span $\mathbb{R}^n$.
(m) The row vectors of A form a basis for $\mathbb{R}^n$.

Standard Basis and Coordinates

The standard unit vectors in $\mathbb{R}^n$ are the n rows and columns of the identity matrix I_n. A **standard basis** for $\mathbb{R}^n$ would be ac collection of n of these vectors, usually denoted $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, ..., $\hat{e}_n$.

Theorem 3.28

Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$ and let $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ be a basis for $\mathcal{S}$. For every vector $\vec{v}$ in $\mathcal{S}$ there is exactly one way to write $\vec{v}$ as a linear combination of the basis vectors in β:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \ldots + c_k\vec{v}_k$$

DEFINITION

These numbers $c_1, c_2, \ldots, c_k$ are called the **coordinates of $\vec{v}$ with respect to β.**

The vector $[\vec{v}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_k \end{bmatrix}$ is known as the **coordinate vector of $\vec{v}$ with respect to β.**

EXAMPLE

Poole, page 209, #49. Show that $(1, 6, 2)$ is in span(β), where $\beta = \{(1, 2, 0), (1, 0, -1)\}$ and find the coordinate vector $[\vec{w}]_\beta$.

CLICKER QUESTION 17.1

The column space of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ consists of

1. all linear combinations of the columns of A.
2. a line in $\mathbb{R}^2$.
3. all multiples of the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
4. All of the above
5. None of the above

CLICKER QUESTION 17.2

Which of the following vectors is in the row space of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$?

1. $\vec{x} = \begin{bmatrix} -2 & 4 \end{bmatrix}$
2. $\vec{x} = \begin{bmatrix} 4 & 8 \end{bmatrix}$
3. $\vec{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}$
4. $\vec{x} = \begin{bmatrix} 8 & 4 \end{bmatrix}$
5. More than one of the above
6. None of the above

CLICKER QUESTION 17.3

The *null space* of a matrix A is the set of vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$. Which of the following vectors is in the null space of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$?

1. $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
2. $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
3. $\vec{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$
4. All of the above
5. None of the above

CLICKER QUESTION 17.4

How many solutions $\vec{x}$ are there to $A\vec{x} = \vec{0}$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$?

1. 0 solutions
2. 1 solution
3. 2 solutions
4. Infinite number of solutions

CLICKER QUESTION 17.5

Which of the following sets of vectors forms a basis (linearly independent spanning set) for $\mathbb{R}^3$?

1. $\{(1,0,0),(0,1,0),(0,0,1)\}$
2. $\{(1,0,1),(1,1,0),(1,1,1)\}$
3. $\{(2,0,0),(0,5,0),(0,0,8)\}$
4. All are bases for $\mathbb{R}^3$.

CLICKER QUESTION 17.6

The row space of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ consists of

1. All linear combinations of the columns of A^T.
2. All multiples of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
3. All linear combinations of the rows of A.
4. All of the above
5. None of the above

CLICKER QUESTION 17.7

Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 2 & -1 & 1 & 1 \end{bmatrix}$. How many vectors are in the nullspace of A?

1. Only one
2. Probably more than one, but it's hard to say how many
3. An infinite number