Linear Systems

Math 214 Spring 2008 — Fowler 309 MWF 9:30 am - 10:25 am

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Class 16: **Wednesday March 5**

SUMMARY Subspaces and Vector Spaces
CURRENT READING Poole 3.5 and 6.1

Summary
A vector space is a collection of vectors where the operations of addition and scalar multiplication follow the "usual" 8 rules. In addition, a vector space is closed under these two operations. A subspace is a vector space which also contains the zero vector.

Homework Assignment
*HW #15: Poole, Section 3.5, 1,3,4,5,***6,7,9,11***,12. EXTRA CREDIT 10.*

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1. Definition of a Vector Space

DEFINITION

A set (or collection) of vectors is *closed under vector addition*: for every two vectors that we pick from the set, their sum is also in the set.

A set (or collection) of vectors is *closed under scalar multiplication*: for every vector that we pick from the set, and for any scalar, their product is in the set.

The above two properties are very special and important. We have a special name for when they are satisfied:

Definition 1. Let V be a set of vectors in $\mathbb{R}^n$ (for some n). V is said to be a **vector space** if it satisfies both of the following conditions:

1. It is **closed under vector addition**: for every $\vec{v}$ and $\vec{w}$ in V, $\vec{v}+\vec{w}$ is in V.
2. It is **closed under scalar multiplication**: for every $\vec{v}$ in V and for every scalar c, $c\vec{v}$ is in V.

Q: What is $\mathbb{R}^n$? It is the set of all n-component vectors.

Example 1. Let V be the set of all vectors of the form $\begin{bmatrix} a \\ 2a+1 \end{bmatrix} \in \mathbb{R}^2$. Let W be the set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix} \in \mathbb{R}^2$. **Q:** Is V a vector space? How about W?

Ans: V no, W yes.

Example 2. Let V be the set of all vectors $(x, y) \in \mathbb{R}^2$ s.t. $x + y = 2$. Let W be the set of all vectors $(x, y) \in \mathbb{R}^2$ s.t. $x + y = 0$.

Is V a vector space? How about W? Ans: V no, W yes. Why?

2. Generalized Vector Spaces

Now we'll see the "full" definition for vector spaces; it's more general and abstract than what we've seen, and very useful in many sciences. (We won't work with it much this semester. If you like it, ask Professor Naimi about the course Math 390 *Linear Spaces.*)

DEFINITION

A (linear) **vector space** is a set V of objects (of *any* kind!) called **vectors**, with two operations, called **vector addition** and **scalar multiplication**, that satisfy the following ten properties (see middle of page 433 of Poole) :

(1) V is closed under vector addition.

(2) V is closed under scalar multiplication.

(3) Addition is commutative.

(4) Addition is associative.

(5) There is a unique additive identity in V.

(6) Each element in V has a unique additive inverse.

(7) The scalar 1 acts as the multiplicative identity.

(8) Scalar multiplication is associative: $c_1(c_2\vec{v}) = (c_1c_2)\vec{v}$.

(9) Scalar multiplication is distributive w.r.t. vector addition: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$.

(10) Scalar multiplication is distributive w.r.t. scalar addition. $(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$.

Example 3. Let M be the set of all 2 by 3 matrices. Let's check that M is a vector space. We call each matrix a vector in M.

GroupWork

Q: Is the set of all 2 by 2 matrices a vector space.? Why or why not?

Q: Is the set of all invertible 2 by 2 matrices a vector space? Why or why not?

Q: Let P_2 = the set of all polynomials of degree 2 or less. Is P_2 a vector space?

3. Subspaces

DEFINITION

Definition 2. A **subspace** of a vector space V is a subset S such that S itself is a vector space under the same operations of addition and scalar multiplication that are already defined on V.

S is said to be a **subspace** of V if and only if S satisfies all of the following conditions:

1. It is **closed under vector addition**: for every $\vec{v}$ and $\vec{w}$ in S, $\vec{v} + \vec{w}$ is in S.
2. It is **closed under scalar multiplication**: for every $\vec{v}$ in S and for every scalar c, $c\vec{v}$ is in S.
3. The zero vector of V is in S

EXAMPLE

1. Let S be the subset of $\mathbb{R}^3$ defined by

$$S = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = x_2 - x_3; x_2, x_3 \in \mathbb{R} \right\}$$

Verify that S is a subspace of $\mathbb{R}^3$. (What geometric object does this correspond to?)

2. Let S be the subset of $\mathbb{R}^3$ defined by

$$S = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} ; x_1, x_2 \in \mathbb{R} \right\}$$

Verify that S is **NOT** a subspace of $\mathbb{R}^3$. (What geometric object does this correspond to?)

3. The subspaces of $\mathbb{R}^3$ are planes or lines through the origin, $\mathbb{R}^3$ itself or $\mathbf{Z}$ (containing only $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$).

Describe the three types of subspaces of $\mathbb{R}^2$.

Theorem 3.19

Let $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ be vectors in $\mathbb{R}^n$. The span$(\vec{v_1}, \vec{v_2}, \ldots \vec{v_n})$ is a subspace of $\mathbb{R}^n$.

EXAMPLE

Let's prove Theorem 3.19.

DEFINITION

Let A be an $m \times n$ matrix. The **row space** of A is the subspace of R^n spanned by the rows of A and is denoted row(A). The **column space** of A is the subspace of R^m spanned by the columns of A and is denoted col(A).

Exercise

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$. Describe row$(A)$ and col(A)