SUMMARY Subspaces and Vector Spaces
CURRENT READING Poole 3.5 and 6.1

Summary
A vector space is a collection of vectors where the operations of addition and scalar multiplication follow the “usual” 8 rules. In addition, a vector space is closed under these two operations. A subspace is a vector space which also contains the zero vector.

Homework Assignment
HW #15: Poole, Section 3.5, 1,3,4,5,6,7,9,11,12. EXTRA CREDIT 10.

1. Definition of a Vector Space
DEFINITION
A set (or collection) of vectors is closed under vector addition: for every two vectors that we pick from the set, their sum is also in the set.

A set (or collection) of vectors is closed under scalar multiplication: for every vector that we pick from the set, and for any scalar, their product is in the set.

The above two properties are very special and important. We have a special name for when they are satisfied:

Definition 1. Let V be a set of vectors in $\mathbb{R}^n$ (for some n). $V$ is said to be a vector space if it satisfies both of the following conditions:

1. It is closed under vector addition: for every $\vec{v}$ and $\vec{w}$ in $V$, $\vec{v} + \vec{w}$ is in $V$.
2. It is closed under scalar multiplication: for every $\vec{v}$ in $V$ and for every scalar $c$, $c\vec{v}$ is in $V$.

Q: What is $\mathbb{R}^n$? It is the set of all n-component vectors.

Example 1. Let $V$ be the set of all vectors of the form $\begin{bmatrix} a \\ 2a + 1 \end{bmatrix} \in \mathbb{R}^2$. Let $W$ be the set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix} \in \mathbb{R}^2$. Q: Is $V$ a vector space? How about $W$?

Ans: $V$ no, $W$ yes.

Example 2. Let $V$ be the set of all vectors $(x, y) \in \mathbb{R}^2$ s.t. $x + y = 2$. Let $W$ be the set of all vectors $(x, y) \in \mathbb{R}^2$ s.t. $x + y = 0$.

2. Generalized Vector Spaces
Now we’ll see the “full” definition for vector spaces; it’s more general and abstract than what we’ve seen, and very useful in many sciences. (We won’t work with it much this semester. If you like it, ask Professor Naimi about the course Math 390 Linear Spaces.)

**DEFINITION**
A (linear) vector space is a set $V$ of objects (of any kind!) called vectors, with two operations, called vector addition and scalar multiplication, that satisfy the following ten properties (see middle of page 433 of Poole):

1. $V$ is closed under vector addition.
2. $V$ is closed under scalar multiplication.
3. Addition is commutative.
4. Addition is associative.
5. There is a unique additive identity in $V$.
6. Each element in $V$ has a unique additive inverse.
7. The scalar 1 acts as the multiplicative identity.
8. Scalar multiplication is associative: $c_1(c_2 \vec{v}) = (c_1c_2)\vec{v}$.
9. Scalar multiplication is distributive w.r.t. vector addition: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$.
10. Scalar multiplication is distributive w.r.t. scalar addition: $(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$.

*Example* 3. Let $M$ be the set of all 2 by 3 matrices. Let’s check that $M$ is a vector space. We call each matrix a vector in $M$.

**GroupWork**
Q: Is the set of all 2 by 2 matrices a vector space? Why or why not?

Q: Is the set of all invertible 2 by 2 matrices a vector space? Why or why not?

Q: Let $P_2$ = the set of all polynomials of degree 2 or less. Is $P_2$ a vector space?
3. Subspaces

**DEFINITION**

Definition 2. A **subspace** of a vector space $V$ is a subset $S$ such that $S$ itself is a vector space under the same operations of addition and scalar multiplication that are already defined on $V$.

$S$ is said to be a **subspace** of $V$ if and only if $S$ satisfies all of the following conditions:

1. It is **closed under vector addition**: for every $\vec{v}$ and $\vec{w}$ in $S$, $\vec{v} + \vec{w}$ is in $S$.
2. It is **closed under scalar multiplication**: for every $\vec{v}$ in $S$ and for every scalar $c$, $c\vec{v}$ is in $S$.
3. The zero vector of $V$ is in $S$

**EXAMPLE**

1. Let $S$ be the subset of $\mathbb{R}^3$ defined by

$$S = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = x_2 - x_3; x_2, x_3 \in \mathbb{R} \right\}$$

Verify that $S$ is a subspace of $\mathbb{R}^3$. (What geometric object does this correspond to?)

2. Let $S$ be the subset of $\mathbb{R}^3$ defined by

$$S = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

Verify that $S$ is **NOT** a subspace of $\mathbb{R}^3$. (What geometric object does this correspond to?)

3. The subspaces of $\mathbb{R}^3$ are planes or lines through the origin, $\mathbb{R}^3$ itself or $\mathbb{Z}$ (containing only $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$).

Describe the three types of subspaces of $\mathbb{R}^2$. 

Theorem 3.19
Let \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) be vectors in \( \mathbb{R}^n \). The span(\( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \)) is a subspace of \( \mathbb{R}^n \).

Example
Let’s prove Theorem 3.19.

Definition
Let \( A \) be an \( m \times n \) matrix. The row space of \( A \) is the subspace of \( \mathbb{R}^n \) spanned by the rows of \( A \) and is denoted \( \text{row}(A) \). The column space of \( A \) is the subspace of \( \mathbb{R}^m \) spanned by the columns of \( A \) and is denoted \( \text{col}(A) \).

Exercise
Let \( A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} \). Describe \( \text{row}(A) \) and \( \text{col}(A) \).