Linear Systems

Math 214 Spring 2008 — Fowler 309 MWF 9:30 am - 10:25 am

 http://faculty.oxy.edu/ron/math/214/08/

Class 12: **Wednesday February 20**

SUMMARY The Inverse Matrix
CURRENT READING Poole 3.3

Summary
We will introduce a very important concept, the Inverse Matrix.

Homework Assignment
HW # 12: Section 3.3 # 2,5, **9,10,19,20,21***, 22,23 EXTRA CREDIT # 13*

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1. Inverse Matrix

Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Write down the product of M and A. That is, MA and AM.

We call M, the matrix which when multiplied by A produces the identity matrix, the **inverse matrix**. It is denoted A^{-1}.

It has the property that $A^{-1}A = AA^{-1} = I$

The factor $ad - bc$ is known as the **determinant** of the matrix A. We will learn more about how to compute determinants and their significance later. However, it is true that if the determinant of a matrix equals zero, then that matrix is NOT invertible, i.e. det $A = 0 \Rightarrow A^{-1}$ doesn't exist. It is also true that if $A^{-1} doesnotexist \Rightarrow \det(A) = 0$.

Theorem 3.6
If A is an invertible matrix, then its inverse A^{-1} is **unique**.

Theorem 3.7
If A is an invertible $n \times n$ matrix, then the system of linear equations given by $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$ for **any** $\vec{b}$ in $\mathbb{R}^n$.

2. Computing Inverses: Gauss-Jordan Elimination

In order to actually generate or find an inverse matrix we use a process called Gauss-Jordan elimination. This is identical to the Gaussian elimination process we already know, except extended.

Consider the system

$$\begin{aligned} 1x + 2y - 1z &= 1 \\ 2x + 2y + 4z &= 3 \\ 1x + 3y - 3z &= 0 \end{aligned}$$

Write down the augmented matrix with the identity matrix as the right hand side.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

We will do Gaussian Elimination on this system until we have produced the identity matrix on the left 3x3 matrix.

3. Properties of Inverses

(1) $(A^{-1})^{-1} = A$

(2) $(AB)^{-1} = B^{-1}A{-}1$

(3) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(4) $(A^{-1})^n = (A^n)^{-1}$ for positive integers n

(5) $(A^{-1})^T = (A^T)^{-1}$

(6) $\frac{1}{c}A^{-1} = (cA)^{-1}$ for non-zero scalars $c \neq 0$

Exercise

Consider $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -7 \\ 2 & 3 \end{bmatrix}$. Show that $A^{-1}B^{-1} = (BA)^{-1}$.

4. Determining Singularity

If the determinant of a coefficient matrix is zero, then the system is singular (no solution or infinite number of solutions) and thus the linear system can not be solved.

$$\det(A) = 0 \longleftrightarrow A^{-1}\text{doesn't exist}$$

So it is NOT always possible to find A^{-1}. A^{-1} exists ONLY IF a $n \times n$ matrix A has rank(n).

5. Using Gauss-Jordan To Solve Linear Systems

Gauss-Jordan takes the augmented matrix $\left[\begin{array}{c} A|I \end{array}\right]$ and converts it into $\left[\begin{array}{c} I|A^{-1} \end{array}\right]$.

Q: What has happened to each block matrix in the augmented matrix?
A: Each block matrix been multiplied by by A^{-1}.

Therefor Gauss-Jordan can also take the matrix $\left[\begin{array}{c} A|I|\vec{b} \end{array}\right]$ and convert into $\left[\begin{array}{c} I|A^{-1}|A^{-1}\vec{b} \end{array}\right]$

Why is this useful?

Gauss-Jordan works by solving n linear systems at once.
For a 3x3 system it is solving $A\vec{x_1} = \vec{e_1}$, $A\vec{x_2} = \vec{e_2}$ and $A\vec{x_3} = \vec{e_3}$
where $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

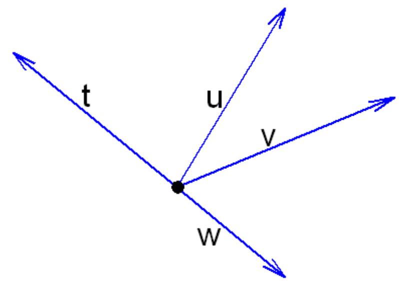
The vectors $\vec{x_1}$, $\vec{x_2}$ and $\vec{x_3}$ which solve the 3 equations above are simply the columns of the inverse matrix.

Example
Consider the system (with $d \neq 0$)

$\left[\begin{array}{ccccccccc} 1 & 1 & 1 & | & 1 & 0 & 0 & 1 \\ 1 & (d+1) & 3 & | & 0 & 1 & 0 & 5 \\ 0 & 2 & d & | & 0 & 0 & 1 & -4 \end{array}\right]$ Let's use Gauss-Jordan to find the solution

CLICKER QUESTION 12.1

Which set of the vectors below is not linearly independent?



A. $\vec{u}, \vec{w}$

B. $\vec{t}, \vec{w}$

C. $\vec{t}, \vec{u}, \vec{v}$

D. All of these sets are linearly independent.

E. More than one of these set is not linearly independent.

CLICKER QUESTION 12.2

Suppose you wish to determine whether a set of vectors is linearly independent.
You form a matrix with those vectors as the columns,
and you calculate its reduced row echelon form

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

What do you decide?

A. These vectors are linearly independent.

B. These vectors are not linearly independent.

CLICKER QUESTION 12.3

To determine whether a set of n vectors from $\mathbb{R}^n$ is independent,
we can form a matrix A whose columns are the vectors in the set
and then put that matrix in reduced row echelon form.
If the vectors are linearly independent,
what will we see in the reduced row echelon form?

A. A row of all zeros.

B. A row that has all zeros except in the last position.

C. A column of all zeros.

D. An identity matrix.

CLICKER QUESTION 12.4

To determine whether a set of fewer than n vectors from $\mathbb{R}^n$ is independent,
we can form a matrix A whose columns are the vectors in the set
and then put that matrix in reduced row echelon form.
If the vectors are linearly independent,
what will we see in the reduced row echelon form?

A. An identity submatrix with zeros below it.

B. A row that has all zeros except in the last position.

C. A column that is not an identity matrix column.

D. A column of all zeros.

CLICKER QUESTION 12.5

If the columns of A are not linearly independent,
how many solutions are there to the system $A\vec{x} = \vec{0}$?

A. 0

B. 1

C. infinite

D. Not enough information is given.