Linear Systems

Math 214 Spring 2008 — Fowler 309 MWF 9:30 am - 10:25 am

 — http://faculty.oxy.edu/ron/math/214/08/

Class 8: **Monday February 11**

SUMMARY Linear Independence and Span
CURRENT READING Poole 2.3

Summary
We will define the important concepts of linear independence, linear dependence, span and spanning set.

Homework Assignment
HW #8: Section 2.3: # 1, **2, 3**, 4, 13, 14, **25, 26**. EXTRA CREDIT: 43.

DEFINTION: homogeneous system

A **homogeneous** linear system is one where the right hand side or constant term in each equation is equal to zero. It looks like $A\vec{x} = \vec{0}$. It always has at least one solution, so it is ALWAYS a **consistent** system. It is consistent because it must possess at least one solution, the trivial solution $\vec{x} = \vec{0}$.

Theorem

A homogeneous system has infinitely many non-zero solutions if it has more variables than equations (i.e. $n > m$). In other words, there are always free variables when the number of variables (n) is greater than the number of equations (m) in a homogeneous linear system.

GroupWork

Can you think of a **geometric** or visual representation of this theorem?

Linear independence

Let's revisit the question of when do we know that a linear combination of a set of vectors equals a given vector.

Suppose you had been told that **any vector** in $\mathbb{R}^2$ can be expressed as a linear combination of $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. How can we **prove** this statement?

How about if we denote "any vector in $\mathbb{R}^2$" to be $\begin{bmatrix} a \\ b \end{bmatrix}$ and attempt to solve the vector equation:

$$x \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

where x and y are unknown scalars that we will try to determine **assuming** a and b exist and putting no conditions on a and b since they can be any real number, so that (a, b) is any point in the plane $\mathbb{R}^2$.

EXAMPLE

Let's form the augmented matrix $[A|\vec{b}]$ where $A = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} a \\ b \end{bmatrix}$, and then apply row reduction.

$$\left[\begin{array}{cc|c} 3 & 1 & a \\ 1 & 5 & b \end{array}\right] \rightarrow$$

Exercise

Show that the solution of the previous question is that $x = \dfrac{5a-b}{14}, \quad y = \dfrac{3b-a}{14}$ so that

$$\frac{5a-b}{14}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{3b-a}{14}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ for ANY } a \text{ and } b$$

Theorem 2.4

The above result can be generalized into a theorem: A system of linear equations with augmented matrix $[A|\vec{b}]$ is **consistent** if and only if $\vec{b}$ is a linear combination of the columns of A.

Discussion

Q: What does "if and only if" mean?

A: It means that the logical implication "goes both ways." In other words, if the statement after "**if and only if**" is true, then it implies the statement BEFORE it is true, **AND** if the statement before the "**if and only if**" is true then that implies the statement after it is true.

Exercise

Is $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ a consistent system?

What does the set of all possible linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ look like?

DEFINITION: span

The **span** of a set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}$ is the set of all linear combinations of those vectors.

$$\text{span}(\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}) = \{c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} \mid c_i \in \mathbb{R}\}$$

A set S of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ in **spans** a vector space if their linear combinations fill the space. (That is, every vector in the space can be written as a linear combination of vectors from the set.) The set $S = \text{span}(\vec{v_1}, \vec{v_2}, \ldots \vec{v_n})$ is called a **spanning set** for the vector space for the vector space.

NOTE

"Span" can be used both as a noun and as a verb. Typically, the vector spaces we are talking about are $\mathbb{R}^n$ but this definition applies to more exotically defined vectors and vector spaces. (We shall see a formal definition of vector space later in the semester.)

Example 1. **Q:** Is (2,3) in the span of $v_1 = (0,1)$ and $v_2 = (1,0)$? **A:** Yes. why?

Q: Is $(2,3) \in \text{span}\{(1,1),(2,2)\})$? **A:** No. Why?

Q: Is $(2,3) \in \text{span}\{(1,1),(1,0),(0,1)\}$? **A:** Yes. Why?

Q: Do $(1,1)$ and $(2,2)$ span $\mathbb{R}^2$? **A:** No. Why?

Q: Do $(1,0)$ and $(0,1)$ span $\mathbb{R}^2$? **A:** Yes. Why?

Q: Describe the span of $\{(1,3)\}$. **A:** The line $y = 3x$ in the xy-plane.

Exercise

What is the span of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$?

Definition: linear independence

A set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ is **linearly independent** provided

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$$

if and only if $c_i = 0$ for $i = 1, 2, \ldots, n$. (The **only** way to combine linearly independent vectors to get the zero vector is to multiply them all by zero scalars.)

Definition: linear dependence

A set of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ is **linearly dependent** provided one of the vectors is a linear combination of the others. (So there is a way to combine **linearly dependent** vectors to get the zero vector by using non-zero scalars.)

Example 2. Let $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Q: Are $\vec{v_1}, \cdots, \vec{v_4}$ linearly independent? **A:** No. Why?

Q: How about $\vec{v_1}, \vec{v_2}, \vec{v_3}$? **A:** Yes. Why?

Q: Describe $W = \text{span}(\vec{v_1}, \cdots, \vec{v_4})$.

NOTE: The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent and span W.

Example 3. Are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ linearly independent? What about $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$? **Explain.**

Theorem 2.6

Let $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ be column vectors in $\mathbb{R}^m$ and let the matrix A be the $m \times n$ matrix with these vectors as columns. The vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ are **linearly dependent** IF AND ONLY IF the homogeneous linear system $A\vec{x} = \vec{0}$ with augmented matrix $[A|\vec{0}]$ has a *non-trivial solution* (i.e. one where $\vec{x} \neq \vec{0}$).

EXAMPLE

Determine whether $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$ are linearly independent or not.

Theorem 2.7

Let $\vec{v_1}, \vec{v_2}, \ldots \vec{v_m}$ be row vectors in $\mathbb{R}^n$ and let the matrix A be the $m \times n$ matrix with these vectors as rows. The vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ are **linearly dependent** IF AND ONLY IF $\text{rank}(A) < m$.

Exercise

Use Theorem 2.7 to determine whether $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 & 3 \end{bmatrix}$ are linearly independent or not. (Look carefully. How are these vectors different from the ones in the EXAMPLE above?)

These results can be summarized in Theorem 2.8.

Theorem 2.8

Any set of m vectors in $\mathbb{R}^n$ is **linearly dependent** IF $m > n$. **Corollary.** It takes at least n vectors to span $\mathbb{R}^n$.

CLICKER QUESTION 8.1

Suppose we have the vectors $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.
Which of the following is true?

A. Every vector in $\mathbb{R}^3$ can be written as a linear combination of these vectors.

B. Some, but not all, vectors in $\mathbb{R}^3$ can be written as a linear combination of these vectors.

C. Every vector in $\mathbb{R}^2$ can be written as a linear combination of these vectors.

D. More than one of the above is true.

E. None of the above are true.

CLICKER QUESTION 8.2

Which of the following vectors can be written
as a linear combination of the vectors (1, 0) and (0, 1)?

A. $(2, 0)$

B. $(-3, 1)$

C. $(0.4, 3.7)$

D. All of the above

CLICKER QUESTION 8.3

How do you describe the set of all
linear combinations of the vectors (1, 0) and (0, 1)?

A. A point

B. A line segment

C. A line

D. $\mathbb{R}^2$

E. $\mathbb{R}^3$

CLICKER QUESTION 8.4

How do you describe the set of all
linear combinations of the vectors (1, 0, 0), (0, 1, 0), (0, 0, 1)?

A. A point

B. A line segment

C. A line

D. $\mathbb{R}^2$

E. $\mathbb{R}^3$

CLICKER QUESTION 8.5

How do you describe the set of all linear combinations of the vectors (1, 2, 0) and (-1, 1, 0)?

A. A point

B. A line

C. A plane

D. $\mathbb{R}^2$

E. $\mathbb{R}^3$

CLICKER QUESTION 8.6

Let z be any vector from $\mathbb{R}^3$. If we have a set V of unknown vectors from $\mathbb{R}^3$, how many vectors must be in V to guarantee that z can be written as a linear combination of the vectors in V?

A. 2

B. 3

C. 4

D. It is not possible to make such a guarantee.

CLICKER QUESTION 10.1

Suppose $\vec{y}$ and $\vec{z}$ are both solutions to $A\vec{x} = \vec{b}$. **True or False** All linear combinations of y and z also solve $A\vec{x} = \vec{b}$. (You should be prepared to support your answer with either a proof or a counterexample.)

A. TRUE.

B. FALSE.

CLICKER QUESTION 10.2

Suppose $\vec{y}$ and $\vec{z}$ are both solutions to $A\vec{x} = \vec{0}$. **True or False** All linear combinations of y and z also solve $A\vec{x} = \vec{0}$. (You should be prepared to support your answer with either a proof or a counterexample.)

A. TRUE.

B. FALSE.

CLICKER QUESTION 10.3

Which set of vectors is linearly independent?

A. $(2, 3), (8, 12)$

B. $(1, 2, 3), (4, 5, 6), (7, 8, 9)$

C. $(-3, 1, 0), (4, 5, 2), (1, 6, 2)$

D. None of these sets are linearly independent.

E. Exactly two of these sets are linearly independent.

F. All of these sets are linearly independent.