Linear Systems

Math 214 Spring 2008 — Fowler 309 MWF 9:30 am - 10:25 am

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Class 7: **Wednesday February 6**

SUMMARY Reduced Row Echelon Form and Rank
CURRENT READING Poole 2.2

Summary
We will discuss the reduced row echelon form of a matrix, sometimes denoted `rref(A)`, and introduce the important concept of rank of a matrix.

Homework Assignment
HW #7: Section 2.2: 1, 2, 3, 4, 5, 6, 7, 8, **22, 23,26**, *27, 28, 29, 36,* **43**. *EXTRA CREDIT # 47. DUE FRI FEB 8.*

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1. Reduced Row Echelon Form

Definition

A matrix is in **reduced row echelon form** if it satisfies the following properties:

1. It is in row echelon form (i.e. all zero rows are at the bottom of the matrix and the first non-zero entry of each non-zero row is in a column to the left of any non-zero leading entry (of a column) in rows below it, forming an echelon or staircase appearance).
2. The leading entry in each non-zero row is 1.
3. Each column containing a leading entry of 1 has zeros everwhere else.

EXAMPLE Consider the following matrices. Are any of them in reduced row echelon form?

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

GroupWork

Find the reduced row echelon matrix for each of the following. (Also note what are the dimensions (i.e. number of rows and number of columns) for each matrix?)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & 4 & 8 \end{bmatrix}$$

2. Definition of Rank

Definition 1. The rank of a $m \times n$ matrix A is the number of non-zero rows in its row echelon form. We call this number $r = \text{rank(A)}$.

By definition $r \leq m$ and $r \leq n$. Also, r is the number of pivots a coefficient matrix has when applying Gaussian Elimination.

THEOREM

When there are more columns than rows in a matrix, i.e. $n > m$ there is ALWAYS atleast one free variable. In other words if A is the $m \times n$ coefficient matrix of a consistent linear system, the number of free variables $= n - r$. This result is known as **the Rank Theorem.**

Exercise

What are the rank of each of the matrices A, B and C given on the other side of this page? How is r related to m and n in each case? Do you notice a pattern?

EXAMPLE

Let's **solve** (find the solution set of) the following linear system of equations

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + z &= 1 \end{aligned}$$

EXAMPLE

Consider $\vec{p} = (1, 0, -1)$, $\vec{q} = (0, 2, 1)$, $\vec{u} = (1, 1, 1)$ and $\vec{v} = (3, -1, -1)$ Do the following lines $\vec{x} = \vec{p} + t\vec{q}$ and $\vec{x} = \vec{u} + t\vec{v}$ intersect? If so, where? Is it possible for lines in $\mathbb{R}^3$ to not intersect and not be parallel?

CLICKER QUESTION 7.2

What is the solution to the system of equations represented with this matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

A. $x = 2,\ y = 3,\ z = 4$

B. $x = -1,\ y = 1,\ z = 1$

C. There are an infinite number of solutions.

D. There is no solution.

E. We can't tell without having the system of equations.

CLICKER QUESTION 7.3

What is the solution to the system of equations represented with this matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

A. $x = 2,\ y = 3,\ z = 4$

B. $x = -1,\ y = 1,\ z = 1$

C. There are an infinite number of solutions.

D. There is no solution.

E. We can't tell without having the system of equations.

CLICKER QUESTION 7.4

What is the solution to the system of equations represented with this matrix?

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A. $x = 2,\ y = 3,\ z = 4$

B. $x = -1,\ y = 1,\ z = 1$

C. There are an infinite number of solutions.

D. There is no solution.

E. We can't tell without having the system of equations.

CLICKER QUESTION 7.5

Suppose we want to graph the equations represented by the rows of the augmented matrix below. What will they look like?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

A. These equations represent two lines that intersect at $x = 2$ and $y = 3$.

B. These equations represent three parallel planes.

C. These equations represent three planes that are not parallel, but which do not share a common point of intersection.

D. These equations cannot be represented geometrically.

CLICKER QUESTION 7.6

Suppose we want to graph the equations represented by the rows of the augmented matrix below. What will they look like?

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A. These equations represent two equations for the same plane.

B. These equations represent three equations for the same plane.

C. These equations represent two planes that have a line of points in common.

D. The intersection of these linear equations is represented by a plane in $\mathbb{R}^3$.

E. These equations cannot be represented geometrically.