Linear Systems

Math 214 Spring 2008 | Fowler 309 MWF 9:30 am - 10:25 am

 | http://faculty.oxy.edu/ron/math/214/08/

Class 6: **Monday February 4**

SUMMARY Solving Linear Systems
CURRENT READING Poole 2.1

OUTLINE
Now that we can visualize and understand the basic nature of linear systems, let's learn some direct techniques for finding solutions of a given linear system.

Homework Assignment
HW #6: Section 2.1 # 1, 3, **10, 11**, *13, 15, 17, 20, 23, 28, 29, 32, 34,* **35, 44**: *DUE WED FEB 6*

Warm-Up
Q: How many solutions is it possible for a linear system to have?
A: ____

1. Elimination

Consider the following linear system of 2 equations in 2 unknowns:

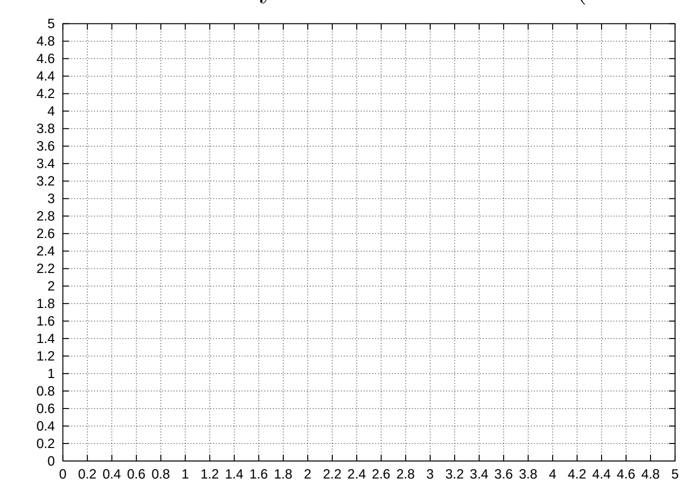
$$\begin{aligned} x + y &= 4 \\ 2x - 3y &= 1 \end{aligned}$$

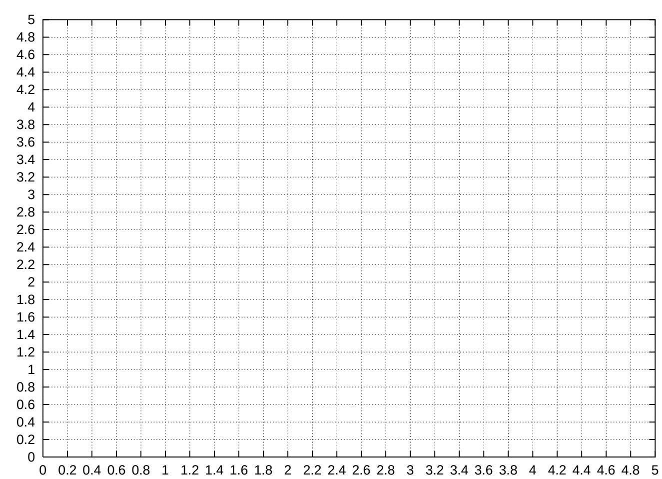
Using algebra one can transform this system into an EQUIVALENT form from which the solution can be easily found by **back-substitution**. This transformation process is called **elimination**.

$$\begin{aligned} x + y &= 4 \\ -5y &= -7 \end{aligned}$$

Using elimination one tries to change the coefficient matrix from $\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ to $\begin{bmatrix} 1 & 1 \\ 0 & -5 \end{bmatrix}$ The transformed coefficient matrix is said to now be **upper triangular**.

On the axes below show how the graphical representations of the equivalent systems change, but the SOLUTION of the system remains the same (obviously).





2. Equivalent Systems of Equations

DEFINITION: **solution**

Suppose we have a system of equations in n variables, $x_1, \cdots, x_n$. An n-component vector $(c_1, \cdots, c_n)$ is said to be a **solution** for the system if substituting c_i for x_i (for all $i = 1, \cdots, n$) simultaneously satisfies all the equations.

DEFINITION: **equivalent system**

Two systems of linear equations are said to be **equivalent** if they have the same solutions (i.e., if any solution of one system is also a solution of the other).

Example 1. **Q:** Are the following systems equivalent? Why or why not?

$$\begin{aligned} x + 2y &= 4 \\ 3x - y &= 5 \end{aligned} \qquad \text{vs.} \qquad 4x + y = 9$$

(eq1 + eq2 in system1 gives the eq in system2)

Ans: __________ **Why?**

Example 2. **Q:** How about the following two systems, are they equivalent?

$$\begin{aligned} x + 2y &= 4 \\ 3x - y &= 5 \end{aligned} \qquad \text{vs.} \qquad \begin{aligned} x + 2y &= 4 \\ 6x - 2y &= 10 \end{aligned}$$

Ans: ____________________ **Why?**

Free variables

Example 3. Solve the following system: $\begin{aligned} x + y + z &= 1 \\ x + 2y + z &= 3 \end{aligned}$

Q: How many solutions does this system have? **Ans.** ____________________

Write down the solution using z as a free variable.

3. Standardizing The Elimination Process

DEFINITION: **row operation**

An **elementary row operation** is any of the following, where c is a nonzero scalar:

1. row$i$ = row$i$ + c(row$k$);
2. row$i$ = c(row$i$);
3. row$i$ = row$k$, and row$k$ = row$i$ (switch rows).

EXAMPLE

Let's write down a example of applying each of the elementary row operations to the matrix $\begin{bmatrix} 4 & 6 & 3 \\ -7 & -1 & 8 \\ 2 & 0 & -1 \end{bmatrix}$.

$\boxed{\text{DEFINITION: } \textbf{pivot}}$

A **pivot** is the first non-zero entry in a row.

$\boxed{\text{DEFINITION: } \textbf{row reduction}}$

The process of applying elementary row operations to a matrix to eliminate coefficients (turn them into zero) is called **row reduction**.

$\boxed{\text{ALGORITHM: } \textbf{Gaussian elimination}}$

1. Find leftmost pivot.
2. If necessary, do row-exchange to "bring up the pivot."
3. (Optional) Divide to make pivot = 1.
4. Make zeros under pivot.
5. Find next leftmost pivot.
6. Go to step 2.

Consider the system

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

which corresponds to the linear system of equations

$$\begin{aligned} 1x + 2y + 3z &= 9 \\ 2x - 1y + 1z &= 8 \\ 3x - 1z &= 3 \end{aligned}$$

We want to get the system into a form which we can solve using back-substitution. To do this we need to identify **pivots**, **multipliers** and look to see if any **row exchanges** will be necessary.

In this system, the coefficient of x in the first row is the **pivot**. We multiply the pivot by the coefficient of x in the second row and subtract rows. The **multiplier** is the number we have to multiply the **pivot** by to eliminate the coefficient we want. This eliminates x in the second row.

In our example above the pivot is __________. The first multiplier is __________.

We repeat the process to eliminate the coefficient of x in the third row. The result is that the first column ends up with zeroes beneath the pivot. **NOTE: We include the right-hand side in our calculations by forming an AUGMENTED COEFFICIENT MATRIX**

$$\begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 2 & -1 & 1 & | & 8 \\ 3 & 0 & -1 & | & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & -5 & -5 & | & -10 \\ 3 & 0 & -1 & | & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & -5 & -5 & | & -10 \\ 0 & -6 & -10 & | & -24 \end{bmatrix}$$

Now we proceed to the second unknown variable y and again look for **pivots** and **multipliers**.

The pivot in the second row is __________ and the multiplier is __________.

If we continue the process of elimination:

$$\begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & -5 & -5 & | & -10 \\ 0 & -6 & -10 & | & -24 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & -5 & -5 & | & -10 \\ 0 & 0 & -4 & | & -12 \end{bmatrix}$$

We have now achieved upper-triangular form so we can solve the system by back-substitution.

You should find that $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is the exact solution to $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$

Congratulations on your first application of the Gaussian Elimination algorithm!

4. Examples of elimination

Exercise

Consider the linear systems. Get them into upper-triangular form. Solve by back-substition.

$$\begin{aligned} 1x + 1y + 1z &= 2 \\ 2x - 2y + 6z &= 7 \\ 1x - 1y + 3z &= 3 \end{aligned}$$

$$\begin{aligned} 0x + 4y &= 2 \\ 1x - 3y &= 1 \end{aligned}$$

$$\begin{aligned} 1x - 1y + 3z &= 3 \\ 2x + 1y + 1z &= 4 \\ 1x - 1y - 1z &= -1 \end{aligned}$$

DEFINITION: **row echelon form**

One goal of row reduction is to transform the augmented coefficient matrix of a linear system into **row echelon form**. A matrix is said to be in **row echelon form** if it satsfies the following properties:

1. Any rows consisting entirely of zeros are at the bottom of the matrix
2. In each nonzero row, the first non-zero entry in that row (known as THE LEADING ENTRY) is always in a column to the left of any leading entries in rows below it.