Linear Systems

Class 2: **Friday January 25**

SUMMARY Lengths and Dot Products
CURRENT READING Poole 1.2

RECALL
Previously we have discussed addition and scalar multiplication of vectors, primarily in the form of linear combinations of vectors. Today we're going to think about *multiplication* of vectors. As with most topics in this course, there's an algebraic view and a geometric (graphical) view of understanding this concept.

Homework Assignment #2
Section 1.2 # **2**,*5, 11,* **17**, *19, 25,* **44**, *46, 47,* **52**: *DUE MON JAN 28*

Consider the vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

1. The Dot Product

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$$

Note this product is a scalar, not a vector. The dot product operation IS commutative. Can you PROVE this?

Also, interestingly $\vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2 = v_1^2 + v_2^2$

the above formula should remind you of the expression for the length or magnitude of a vector $\vec{v}$, which is usually denoted $||\vec{v}||$

So,

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$$

2. Normalization
Oftentimes we want to work with vectors of unit length. These vectors are called *normalized*.

Suppose $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ How can we normalize $\vec{w}$ so that it has the same direction, but magnitude 1?

In general, a vector $\vec{v}$ is normalized by ______________________________________.

3. Orthogonality.

Suppose $\vec{a} \cdot \vec{b} = 0$ what can we say about $\vec{a}$ and $\vec{b}$?

Consider $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

AND

$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

AND

$\vec{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Do you notice a pattern among these vectors? What properties do the vector pairs share?

4. Angle Between Vectors.

Consider the vectors $\vec{v} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$

$\vec{v} \cdot \vec{w} =$ ____.

Can you rewrite this formula using a trigonometric identity?

Draw $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ on an axes.

1. What is the dot product between them?
2. What is the angle between them?
3. What is the angle between $\vec{a}$ and $2\vec{b}$?

5. Angle Formula. For any vectors $\vec{v}$ and $\vec{w}$ where θ is the smaller angle between them,

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||}$$

How can we use this formula to confirm our understanding of orthogonality? (i.e. what happens when the dot product between two vectors is zero?)

6. Projection. For any vectors $\vec{u}$ and $\vec{v}$ where $\vec{u} \neq 0$ then **the projection of $\vec{v}$ onto $\vec{u}$** is the vector $\mathrm{proj}_{\vec{u}}(\vec{v})$ defined by:

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)\vec{u}$$