

Goal: To project $\vec{v} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$ onto the orthogonal complement of the plane $x - y + 3z = 0$ (denoted by \mathcal{W}) in \mathbb{R}^3 .

(a) (2 points). If the set of points (x, y, z) in \mathbb{R}^3 are represented as the vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the set of points on the plane $x - y + 3z = 0$ can be represented as the vectors found in the nullspace of what matrix? (Re-write $x - y + 3z = 0$ as $A\vec{x} = 0$ and identify A and list its dimensions).

$$\begin{bmatrix} 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad A \text{ is } 1 \times 3 \text{ (1 row, 3 cols)}$$

$$A = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$$

(b) (2 points). Since the plane $x - y + 3z = 0$ is a 2-dimensional object and given that the rank of the matrix A is 1, write down a basis for $\text{null}(A)$ which contains 2 vectors.

$$\text{null}(A) = \begin{pmatrix} -3z + y \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) (2 points). Using the Rank Theorem, write down a basis for the orthogonal complement of the nullspace of A . (HINT: what is the dimension of this orthogonal complement?)

The orthogonal complement must be orthogonal to $\text{null}(A)$, i.e. $\text{row}(A)$.

$$(\text{null}(A))^\perp = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\} = \text{row}(A)$$

(d) (2 points). Your answer in (c) is a basis for the orthogonal complement of the nullspace of A , in other words \mathcal{W}^\perp . Find $\vec{w}_1 = \text{proj}_{\mathcal{W}^\perp}(\vec{v})$

$$\vec{w}_1 = \frac{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \frac{-4}{11} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4/11 \\ 4/11 \\ -12/11 \end{pmatrix}$$

(d) (2 points). Use your answer to find $\vec{w}_2 = \text{proj}_{\mathcal{W}}(\vec{v})$. What properties do \vec{w}_1 and \vec{w}_2 have that you can check to confirm your answers?

$$\vec{w}_2 + \vec{w}_1 = \vec{v} \quad \text{and} \quad \vec{w}_1 \cdot \vec{w}_2 = 0$$

$$\vec{w}_2 = \vec{v} - \vec{w}_1 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} -4/11 \\ 4/11 \\ -12/11 \end{pmatrix} = \begin{pmatrix} 15/11 \\ 51/11 \\ 12/11 \end{pmatrix}$$

$$\begin{pmatrix} 15/11 \\ 51/11 \\ 12/11 \end{pmatrix} \cdot \begin{pmatrix} -4/11 \\ 4/11 \\ -12/11 \end{pmatrix} = \frac{-60 + 204 - 144}{121} = 0$$