Math 214 Spring 2008 **SHOW ALL YOUR WORK** Quiz Eight

1. Consider the matrix $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$. We want to obtain a value for $A^\infty = \lim_{n\to\infty} A^n$.

a. *(4 points).* Find the eigenvalues and eigenvectors of A.

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

$$(\lambda + \tfrac{1}{2})(\lambda - 1) = 0$$

$$\lambda = 1, -\tfrac{1}{2}$$

$\lambda = 1$

$$\text{null}(A - I) = \begin{pmatrix} -1/2 & 1/2 \\ 1 & -1 \end{pmatrix} \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 - x_2 = 0, \ x_2 \text{ free}$$

$$E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \qquad E_{-1/2} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

$\lambda = -1/2$

$$\text{null}(A + \tfrac{1}{2}I) = \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 1 & 1/2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad 2x_1 + x_2 = 0, \ x_2 \text{ free}$$

b. *(2 points)* Show that $AS = S\Lambda$ or $A = S\Lambda S^{-1}$, where the columns of S are formed by the eigenvectors of A and Λ is a diagonal matrix with the eigenvalues of A along the diagonal and zeroes elsewhere.

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} \qquad S\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix}$$

$$AS = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix} = S\Lambda$$

$$S^{-1} = \frac{1}{-3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix}$$

c. *(2 points).* Compute $A^n = S\Lambda^n S^{-1}$.

$$A^n = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}^n \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (-1/2)^n \\ 1 & -2(-\frac{1}{2})^n \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 2/3 + \frac{1}{3}(-\frac{1}{2})^n & \frac{1}{3} - \frac{1}{3}(-\frac{1}{2})^n \\ \frac{2}{3} - \frac{2}{3}(-\frac{1}{2})^n & \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^n \end{pmatrix}$$

d. *(2 points).* Use your answer from **c** to show that $A^\infty = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$.

$$\lim_{n\to\infty} A^n = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} \qquad \text{since } \lim_{n\to\infty}\left(-\tfrac{1}{2}\right)^n = 0$$