

1. Consider the system of equations $\begin{bmatrix} b & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$ where b is an unknown parameter.

a. (4 points). If $b = 2$ what are the column space and the nullspace of the coefficient matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$? Write expressions describing these subspaces. Also give the rank of A , the dimension of the nullspace and the dimension of the column space.

$$\text{col}(A) = \text{span} \left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\} = \left\{ \vec{x} = c \begin{pmatrix} 1 \\ 2 \end{pmatrix}, c \in \mathbb{R} \right\} \subset \mathbb{R}^2$$

$$\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 6 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$x + \frac{3}{2}y = 0$$

y free

$$\vec{x} = \begin{pmatrix} -3/2 y \\ y \end{pmatrix} = y \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$

$$\text{null}(A) = \left\{ \vec{x} = c \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}, c \in \mathbb{R} \right\} \subset \mathbb{R}^2$$

$$\text{rank}(A) = 1 \quad \dim(\text{null}(A)) = 1 \quad \dim(\text{col}(A)) = 1$$

b. (4 points). If $b = 1$ how do the column space and nullspace of the coefficient matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$ change from your answer in (a)? Write down the rank of A , the dimension of the nullspace and the dimension of the column space.

$$b=1, \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 4 & 6 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -6 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) = \text{rref}(A)$$

$$\text{rank}(A) = 2$$

$$\text{col}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\} = \mathbb{R}^2$$

$$\text{null}(A) = \{ \vec{0} \}$$

$$\dim(\text{col}(A)) = 2 = \text{rank}$$

$$\dim(\text{null}(A)) = 0$$

c. (2 points). How does the rank of the coefficient matrix $\begin{bmatrix} b & 3 \\ 4 & 6 \end{bmatrix}$ depend on the value of b ?

Use this information to determine for what values of b the system has 1 unique solution and explain how you know the solution will be unique for these values of b .

$$\text{rank}(A) = \begin{cases} 2 & b \neq 2 \quad \& \text{1 unique solution} \\ 1 & b = 2 \quad \& \text{no solution} \end{cases}$$