HW 7

2.2 Direct Methods for Solving Linear Systems

1. No, this matrix is not in row echelon form. Why not? Give at least one reason. The leading entry in row 3 appears to the left of the leading entry in row 2.

2. This matrix is in row echelon form, but not reduced row echelon form. Why not?. There are many reasons. For example, the leading entry in row 1 is 7 not 1.

3. This matrix is in row echelon form, and also reduced row echelon form. Why is the 3 okay? The 3 occurs in a column that does not contain a leading 1.

4. This matrix is in row echelon form, and also reduced row echelon form. Why are the 0s okay? All three rows are zero, so no leading 1s are required.

5. No, this matrix is not in row echelon form. Why not? Give a reason. The row of all zeroes is not at the bottom.

6. No, this matrix is not in row echelon form. Why not? Give a reason. The leading entry in row 3 appears to the left of the leading entry in row 2.

7. No, this matrix is not in row echelon form. Why not? Give a reason. The leading entry in row 2 appears underneath the leading entry in row 1.

8. This matrix is in row echelon form, but not reduced row echelon form. Why not? The leading entry in row 4 is not a 1. Could we have given another reason?

9. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (b) ... $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

10. (a) $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 4 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$ ... (b) ... $\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

11. (a) $\begin{bmatrix} 3 & 5 \\ 5 & -2 \\ 2 & 4 \end{bmatrix} \xrightarrow[R_3 - \frac{2}{3}R_1]{R_2 - \frac{5}{3}R_1} \begin{bmatrix} 3 & 5 \\ 0 & -\frac{31}{3} \\ 0 & \frac{2}{3} \end{bmatrix} \xrightarrow[\frac{3}{2}R_3]{-\frac{3}{31}R_2} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) Continuing from (a): $\begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 5R_2} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

12. (a) $\begin{bmatrix} 2 & -4 & -2 & 6 \\ 3 & 1 & 6 & 6 \end{bmatrix} \xrightarrow{-2R_2} \begin{bmatrix} 2 & -4 & -2 & 6 \\ -6 & -2 & -12 & -12 \end{bmatrix} \xrightarrow{R_2 + 3R_1} \begin{bmatrix} 2 & -4 & -2 & 6 \\ 0 & -14 & -18 & 6 \end{bmatrix}$.

(b) ... from (a): $\begin{bmatrix} 2 & -4 & -2 & 6 \\ 0 & -14 & -18 & 6 \end{bmatrix} \xrightarrow[-\frac{1}{14}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & \frac{9}{7} & -\frac{3}{7} \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{15}{7} \\ 0 & 1 & \frac{9}{7} & -\frac{3}{7} \end{bmatrix}$.

19. Performing $R_2 + R_1$ and $R_1 + R_2$ does *not* leave rows 1 and 2 identical.
After performing $R_2 + R_1$ the second row is now $R_2' = R_2 + R_1$.
So $R_1 + R_2$ is actually $R_1 + R_2' = R_1 + (R_2 + R_1) = 2R_1 + R_2$.
Performing $R_2 + R_1$ and $R_1 + R_2$ simultaneously annuls their linearity.

20. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} x_1 \\ x_2 + x_1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} -x_2 \\ x_2 + x_1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$.
The net effect is to interchange the first and second rows.

21. Our first task is to show that $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{3R_2-2R_1} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}$ is *not* an elementary row operation.
Compare $3R_2 - 2R_1$ to the elementary row operations $R_i \leftrightarrow R_j$, kR_i, $R_i + kR_j$.
Clearly, $3R_2 - 2R_1$ is a combination of kR_i and $R_i + kR_j$ done at the same time.
Performing row operations simultaneously annuls their linearity.

One way to achieve the result is: $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2-\frac{2}{3}R_1} \begin{bmatrix} 3 & 1 \\ 0 & \frac{10}{3} \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}$.

22. We must show that we can create a 1 in row 1, column 1 using $R_i \leftrightarrow R_j$, kR_i, or $R_i + kR_j$.
$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{2}{3} \\ 1 & 4 \end{bmatrix}$. $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & -6 \\ 1 & 4 \end{bmatrix}$.
$R_i \leftrightarrow R_j$ is the most direct. That is, it requires the fewest operations.
kR_i requires fewer operations than $R_i + kR_j$, but $R_i + kR_j$ gives integer results.

23. Since **rank** = the number of nonzero rows in the row echelon form of a matrix, before we answer we should put each of the matrices into row echelon form.

(1) Since this matrix A is not in its row echelon form B, we must row reduce A first.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$ So, rank A = the number of nonzero rows in $B = 3$.

(2) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has two nonzero rows, rank $A = 2$.

(3) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has two nonzero rows, rank $A = 2$.

(4) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has no nonzero rows, rank $A = 0$.

(5) Since this matrix A is not in its row echelon form B, we must row reduce A first.

$$\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ So, rank $A = 2$.

(6) Since this matrix A is not in its row echelon form B, we must row reduce A first.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ So, rank $A = 3$.

(7) Since this matrix A is not in its row echelon form B, we must row reduce A first.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 - \frac{1}{2}R_1 + \frac{1}{2}R_2 \\ R_4 - R_1 + R_2 + 2R_3}]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$ So, rank $A = 3$.

(8) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has 3 nonzero rows, rank $A = 3$.

24. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

25. We have the following system of equations: $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \end{bmatrix}$.

We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+3R_3} \left[\begin{array}{ccc|c} 13 & -1 & 0 & 21 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right] \xrightarrow[-R_3]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 13 & -1 & 0 & 21 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$\xrightarrow[-R_2]{R_1-R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ -13 & 1 & 0 & -21 \\ 0 & -1 & 1 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -13 & 1 & 0 & -21 \\ 0 & -1 & 1 & -4 \end{array}\right] \xrightarrow{R_2+13R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$. So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

26. We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 3 & 1 & 5 \\ 3 & 1 & 7 & 2 \end{array}\right] \xrightarrow{R_3-5R_1-2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 3 & 1 & 5 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

The third row is equivalent to the equation $0 = -8$ which clearly has no solution. Therefore, the system is inconsistent.

Does $R_3 = 5R_1 + 2R_2$ (excluding constants) cause the system to be inconsistent?

27. We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow{R_3+8R_1+10R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 = -8R_1 - 10R_2$ (excluding constants) does not cause a problem here? Why?
Since the system is homogeneous (all constants $= 0$), the system has at least one solution.

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The third row of $0 = 0$ tells us that $x_3 = t$ is a free variable.
Back substituting, we have $x_2 + t = 0 \Rightarrow x_2 = -t$ and $x_1 + t = 0 \Rightarrow x_1 = -t$.

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ or equivalently $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

28. From the beginning, we know this system has infinitely many solutions. Why? Because this system has 4 variables and only 3 equations they have to satisfy.

We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 4 & 0 \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & 1 & 4 & \frac{1}{2} \end{array}\right].$$

The fact that $4 - 3 = 1$ tells us that $z = t$ is a free variable.
So $y + 4t = \frac{1}{2} \Rightarrow y = \frac{1}{2} - 4t$, $x + 2t = \frac{1}{2} \Rightarrow x = \frac{1}{2} - 2t$, and $w + t = \frac{1}{2} \Rightarrow w = \frac{1}{2} - t$.

So, the solution is $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ -4 \\ 1 \end{bmatrix}$.

29. Note that there are 3 equations but only 2 variables to satisfy them.
It is helpful, therefore, to begin by noting $R_3 = 9R_1 - 4R_2$.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}\right] \xrightarrow{R_3 - 9R_1 + 4R_2} \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \cdots \longrightarrow \cdots \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

So, the solution is $\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

30. From the beginning, we know this system has infinitely many solutions. Why? Because this system has 4 variables and only 3 equations they have to satisfy.

We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since rank $A = 2$ and $4 - 2 = 2$, we get 2 free variables: $x_2 = s$ and $x_4 = t$.
Back substituting, we get $x_4 = t$, $x_3 = 1 + 2t$, $x_2 = s$, and $x_1 = -2 + 3s$.

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.

.) Begin by thinking of this system as $[A|\mathbf{x}]$, then determine rank A by inspection.
Mentally performing $R_3 - 2R_2$ implies the equation $0 = 2$.
This equation makes it obvious that this system has no solution.

Note: $R_3 = 2R_2$ implies rank $A = 2$. How does that relate to our answer?

'. Since this system has 4 variables and at most 3 equations,
it has infinitely many solutions. Why? There is at least one free variable.

3. Since this system has 5 variables and at most 3 equations,
it has infinitely many solutions. Why? There are at least two free variables.

9. We need only show that the condition $ad - bc \neq 0$ implies that rank $A = 2$. Why?
If rank $A = 2$, there are $2 - 2 = 0$ free variables so the system has a unique solution.

Case 1: $a = 0$, which implies both $b \neq 0$ and $c \neq 0$. Why? Because $0d - bc = -bc \neq 0$.

Row reduce A: $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$.

A is now in row echelon form with 2 nonzero rows. Therefore, rank $A = 2$.

Case 2: $c = 0$, which implies both $a \neq 0$ and $d \neq 0$. Why? Because $ad - b0 = ad \neq 0$.

Row reduce A: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

A is now in row echelon form with 2 nonzero rows. Therefore, rank $A = 2$.

Case 3: $a \neq 0$ and $c \neq 0$.

Row reduce A: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[aR_2]{cR_1} \begin{bmatrix} ac & bc \\ ac & ad \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} ac & bc \\ 0 & ad - bc \end{bmatrix}$.

A is now in row echelon form with 2 nonzero rows. Therefore, rank $A = 2$.

40. First row reduce th

$\begin{bmatrix} k & 2 & 3 \\ 2 & -4 & -6 \end{bmatrix}$

(a) There are no
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$2 + 2k = 0 \Rightarrow$
But $k = -1 \Rightarrow$

(b) The system ha
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41. First row reduce th

$\begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}$

(a) When $k = -1$
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$1 - k^2 = 0 \Rightarrow$
And $k = -1$

(b) When $k \neq \pm$
From (a), we

(c) When $k = 1$,
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From (a), we

42. First row reduce t

$\begin{bmatrix} 1 & - \\ 1 & \\ 2 & - \end{bmatrix}$

(a) When $k \neq -$
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$k^2 - k - 2 =$

(b) This system
Since rank A

(c) When $k = -$
The system
From (a), w

43. First row reduce the system $[A|\mathrm{x}]$ and then answers parts (a), (b), and (c).

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & k^2+k-2 & k+2 \end{array}\right]$$

(a) When $k = 1$, this system has no solution. Why?
The system has no solution when A has a zero row with corresponding constant $\neq 0$.
$k^2+k-2=0 \Rightarrow k=1$ makes the bottom row 0, but the constant $k+2=1+2=3\neq 0$.

(b) When $k \neq 1, -2$, this system has a unique solution. Why?
When $k \neq 1, -2$, rank $A = 3$. So, there are $3-3=0$ free variables.

(c) When $k = -2$, this system has infinitely many solutions.
The system has infinitely many solutions when A has a zero row with constant $= 0$.
$k^2+k-2=0 \Rightarrow k=-2$ makes the bottom row 0 and the constant $k+2=-2+2=0$.

44. (a) The following system of n equations has infinitely many solutions:

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0 \\ 2x_1 + 2x_2 + \cdots + 2x_n &= 0 \\ &\vdots \\ nx_1 + nx_2 + \cdots + nx_n &= 0 \end{aligned}$$

Likewise, the following system of $n+1$ equations has infinitely many solutions:

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0 \\ 2x_1 + 2x_2 + \cdots + 2x_n &= 0 \\ &\vdots \\ nx_1 + nx_2 + \cdots + nx_n &= 0 \\ (n+1)\,x_1 + (n+2)\,x_2 + \cdots + (n+1)\,x_n &= 0 \end{aligned}$$

(b) The system of n equations $x_1 = 0$, $x_2 = 0$, …, $x_n = 0$ has the unique solution $x_i = 0$.
as does the system of $2n$ equations $x_1 = 0$, $2x_1 = 0$ …, $x_n = 0$, $2x_n = 0$.

45. As in Example 2.
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46. As in Example 2
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47. When looking for examples, begin with familiar planes like $x = 0$, $y = 0$, and $z = 0$.

(a) Let's start with $x = 0$ and $y = 0$. These planes obviously intersect in the z-axis. Why?

As in Exercise 45: $\begin{array}{rcrcrcl} x & + & 0y & + & 0z & = & 0 \\ 0x & + & y & + & 0z & = & 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow x = 0, y = 0, z = t \Rightarrow$

The line of intersection of $x = 0$ and $y = 0$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, the z-axis.

All we need is one other plane that passes through the z-axis to complete our example.
It may help to sketch $\mathbb{R}^2$ and look for a line that passes through the origin.
One such line is $x = y$ which corresponds to the plane $x - y = 0$ in $\mathbb{R}^3$.

Sketch these three planes in $\mathbb{R}^3$ to confirm they intersect in the z-axis.

Q: How do we confirm these three planes intersect in the z-axis algebraically?
A: Check the intersection between $x = 0$ and $x - y = 0$. Why is that enough?

As in Exercise 45: $\begin{array}{rcrcrcl} x & + & 0y & + & 0z & = & 0 \\ x & - & y & + & 0z & = & 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow x = y = 0, \text{ and } z = t \Rightarrow$

The line of intersection of $x = 0$ and $x - y = 0$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Q: Start with $y = 0$ and $z = 0$ and then $x = 1$ and $y = 1$. Is there a pattern?

(b) Begin with $x = 0$ and $y = 0$. We need one plane that crosses across both of these.
It may help to visualize $\mathbb{R}^2$ and look for a line that cuts across the first quadrant.
It is obvious that the plane $x + y = 1$ will complete the example?

Sketch these three planes in $\mathbb{R}^3$ to confirm they intersect in pairs.
For example, $x = 0$ and $x + y = 1$ intersect in the line $[x, y, z] = [0, 1, 0] + t[0, 0, 1]$.

As in Exercise 45: $\begin{array}{rcrcrcl} x & + & 0y & + & 0z & = & 0 \\ x & + & y & + & 0z & = & 1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow x = 0, y = 1, \text{ and } z = t \Rightarrow$

The line of intersection of $x = 0$ and $x + y = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) An obvious example is $x = 0$, $x = 1$, and $y = 0$. Why?
The normal vector for $x = 0$ and $x = 1$ is $[1, 0, 0]$
while the normal vector for $y = 0$ is $[0, 1, 0]$.

(d) The most obvious example is $x = 0$, $y = 0$, and $z = 0$.
Note that any example of $x = a$, $y = a$, and $z = a$ will work.
Are there any other obvious pattern examples that will work?

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