Introduction to Systems of Linear Equations

Following Example 2.1 and justify our assertion by applying the definition of linear. 
\[ ax + (b) z = 0 \] is linear because power of \( z \) is 1 and \( a, b \) are constants.

We follow Example 2.1 and justify our assertion by applying the definition of linear. 
\[ x^2 + y^2 + z^2 = 1 \] is not linear because \( x, y, z \) occur to the power 2.

\[ x^{-1} + 7y + z = \sin \frac{\pi}{5} \] is not linear because \( x \) occurs to the power -1.

\[ 2x - xy - 5z = 0 \] is not linear because the product \( xy \) is of degree 2.

\[ 3 \cos x - 4y + z = \sqrt{3} \] is not linear because \( \cos x \) is not linear.

\[ (\cos 3)x - 4y + z = \sqrt{3} \] is linear because \( \cos 3 \) and \( \sqrt{3} \) are constants.

As in Section 1.3, we put the equation of this line into general form \( ax + by = c \).

\[ 2x + y = 7 - 3y \] is equivalent to \( 2x + 4y = 7 \) after adding \( 3y \) to both sides.

**Note:** When the equation is linear there is no restriction on \( x \) and \( y \). Why?

8. We begin by determining the restrictions on the variables \( x \) and \( y \).

Typical sources are 1) division, 2) square roots, and 3) domains (like \( \log x \Rightarrow x > 0 \)).

   **Step 1.** Determine restriction \textit{type}. With \( \frac{x^2 - y^2}{x - y} = 1 \), it is division.

   **Step 2.** Set the denominator equal to zero to determine the restriction.

   We have \( x - y = 0 \Rightarrow x = y \). So, the restriction is \( x \neq y \).

   **Step 3.** Simplify the given equation using algebra.

   \[ \frac{x^2 - y^2}{x - y} = 1 \Rightarrow \frac{(x - y)(x + y)}{x - y} = 1 \text{-cancel} \Rightarrow x + y = 1. \]

   **Note:** This tells us the given function is equivalent to the line \( x + y = 1 \) provided \( x \neq y \).

9. We begin by determining the restrictions on the variables \( x \) and \( y \).

   Typical sources are 1) division, 2) square roots, and 3) domains (like \( \log x \Rightarrow x > 0 \)).

   **Step 1.** Determine restriction \textit{type}. With \( \frac{1}{x} + \frac{1}{y} = \frac{4}{xy} \), it is division.

   **Step 2.** Set the denominators equal to zero to determine the restriction.

   We have \( x = 0, y = 0 \), and \( xy = 0 \). So, the restriction is \( x, y \neq 0 \).

   **Step 3.** Simplify the given equation using algebra.

   \[ \frac{1}{x} + \frac{1}{y} = \frac{4}{xy} \text{ common denominator} \Rightarrow \frac{y}{xy} + \frac{x}{xy} = \frac{4}{xy} \text{ multiply both sides} \Rightarrow x + y = 4. \]

   **Note:** This tells us the given function is equivalent to the line \( x + y = 4 \) provided \( x, y \neq 0 \).
We begin by determining the restrictions on the variables \(x\) and \(y\).

Typical sources are 1) division, 2) square roots, and 3) domains (like \(\log x \Rightarrow x > 0\)).

Step 1. Determine restriction type. With \(\log_{10} x - \log_{10} y = 2\), it is logarithms.

Step 2. Apply the domain restrictions to determine the overall restriction.

In this case, we have the overall restriction of \(x > 0\) and \(y > 0\).

Step 3. Simplify the given equation using algebra.

\[
\log_{10} x - \log_{10} y = 2 \quad \Rightarrow \quad \log_{10} \frac{x}{y} = 2 \quad \Rightarrow \quad \log_{10} \frac{x}{y} = \log_{10} 10^2
\]

cancel and simplify \(x\) put in general form

\[
\frac{x}{y} = 100 \quad \Rightarrow \quad x - 100y = 0.
\]

**Note:** This tells us the given function is equivalent to the line \(x - 100y = 0\) provided \(x, y > 0\).

11. As in Example 2.2(a), we set \(x = t\) and solve for \(y\).

Setting \(x = t\) in \(3x - 6y = 0\) gives us \(3t - 6y = 0\). Solving for \(y\) yields \(6y = 3t \Rightarrow y = \frac{1}{2}t\).

So, we see the complete set of solutions can be written in the parametric form \([t, \frac{1}{2}t]\).

**Note:** We could have set \(y = t\) to get \(3x - 6t = 0\) and solved for \(x\) so \(x = 2t\) and \([2t, t]\).

12. As in Example 2.2(a), we set \(x_1 = t\) and solve for \(x_2\).

Setting \(x_1 = t\) yields \(2t + 3x_2 = 5\). Solving for \(x_2\) yields \(3x_2 = 5 - 2t \Rightarrow x_2 = \frac{5}{3} - \frac{2}{3}t\).

So, a complete set of solutions written in parametric form is \([t, \frac{5}{3} - \frac{2}{3}t]\).

**Note:** We could have set \(x_2 = t\) and solved for \(x_1\) to get the parametric form \([\frac{5}{3} - \frac{2}{3}t, t]\).

13. As in Example 2.2(b), we set \(y = s\), \(z = t\) and solve for \(x\). (Why is this a good choice?)

This substitution yields \(x + 2s + 3t = 4\). Solving for \(x\) yields \(x = 4 - 2s - 3t\).

So, a complete set of solutions written in parametric form is \([4 - 2s - 3t, s, t]\).

14. As in Example 2.2(b), we set \(x_1 = s\), \(x_2 = t\) and solve for \(x_3\).

This substitution yields \(4s + 3t + 2x_3 = 1\). Solving for \(x_3\) yields \(x_3 = \frac{1}{2} - 2s - \frac{3}{2}t\).

So, a complete set of solutions written in parametric form is \([s, \frac{1}{2} - 2s - \frac{3}{2}t, t]\).
Introduction to Systems of Linear Equations

Augmented matrix

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & 1 & 2 \\
1 & 1 & 2 & 1 & -1 & 4 \\
0 & 1 & 0 & 2 & 3 & 0
\end{bmatrix}
\]

becomes

\[
a - b + 3d + e = 2
\]

\[
b + 2c + d - e = 4
\]

In Example 2.4(a), we add \((x - y) + (2x + y) = 0 + 3\) to get \(3x = 3 \Rightarrow x = 1\) and \(y = 1\).

Quick check confirms that \([1, 1]\) is indeed the unique solution of the system.

Example 2.4(c)

After Example 2.6, we row reduce the augmented matrix from Exercise 28.

\[
\begin{bmatrix}
2 & 3 & -1 & 1 \\
1 & 0 & 1 & 0 \\
-1 & 2 & -2 & 0
\end{bmatrix}
\]

Row operations:

\[
R_1 \rightarrow R_2
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
2 & 3 & -1 & 1 \\
-1 & 2 & -2 & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 + R_2
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
2 & 3 & -1 & 1 \\
0 & 1 & -1 & 1/3
\end{bmatrix}
\]

\[
\frac{1}{3} R_3 \rightarrow R_3
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
2 & 3 & -1 & 1 \\
0 & 1 & -1 & 1/3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 2/3 \\
0 & 1 & -1 & 1/3 \\
0 & 0 & 1 & -2/3
\end{bmatrix}
\]

\[
x_1 = \frac{2}{3}, x_2 = -\frac{1}{3}, \text{ and } x_3 = -\frac{2}{3}.
\]

So the solution is \([x_1, x_2, x_3] = [\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}]\).

As shown after Example 2.6, we row reduce the augmented matrix from Exercise 29.

\[
\begin{bmatrix}
1 & 5 & -1 \\
-1 & 0 & 1 \\
2 & 4 & 1
\end{bmatrix}
\]

Row operations:

\[
R_2 \rightarrow R_2 + R_1
\]

\[
\begin{bmatrix}
1 & 5 & -1 \\
0 & 6 & -6 \\
2 & 4 & 1
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 + R_2
\]

\[
\begin{bmatrix}
1 & 5 & -1 \\
0 & 6 & -6 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
y = -1 \text{ and } z = -1 - 5(-1) = 4,
\]

so the solution is \([x, y] = [4, -1]\).

As shown after Example 2.6, we row reduce the augmented matrix from Exercise 30.

\[
\begin{bmatrix}
1 & -2 & 0 & 1 & 2 \\
1 & 1 & -1 & 3 & 1 \\
2 & 4 & 1 & 2
\end{bmatrix}
\]

Row operations:

\[
R_1 \rightarrow R_1 + R_2
\]

\[
\begin{bmatrix}
1 & -2 & 0 & 1 & 2 \\
0 & -1 & 1 & -2 & 3 \\
2 & 4 & 1 & 2
\end{bmatrix}
\]

\[
d = t, c = s, b = -3 - s - 2t, \text{ and } a = 2 + 2(-3 - s - 2t) - t = -4 - 2s - 5t,
\]

so the solution is \([a, b, c, d] = [-4 - 2s - 5t, -3 - s - 2t, s, t]\).

As shown after Example 2.6, we row reduce the augmented matrix from Exercise 31.

\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & -1 & 0 & 1 \\
2 & -1 & 1 & 1
\end{bmatrix}
\]

Row operations:

\[
R_1 \rightarrow R_1 + R_2
\]

\[
\begin{bmatrix}
2 & -1 & 1 & 1 \\
2 & -1 & 0 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

\[
R_2 \rightarrow R_2 - R_1
\]

\[
\begin{bmatrix}
2 & -1 & 1 & 1 \\
0 & -1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 + R_1
\]

\[
\begin{bmatrix}
2 & -1 & 1 & 1 \\
0 & -1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

\[
\Rightarrow 0 = 0 \Rightarrow \text{No solution.}
\]

As shown after Example 2.6, we row reduce the augmented matrix from Exercise 32.

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & 1 & 2 \\
1 & 1 & 2 & 1 & -1 & 4 \\
0 & 1 & 0 & 2 & 3 & 0
\end{bmatrix}
\]

Row operations:

\[
R_2 \rightarrow R_2 - R_1
\]

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & 1 & 2 \\
0 & 2 & 2 & -2 & 2 \\
0 & 2 & 0 & 4 & 6 & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 - R_2
\]

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & 1 & 2 \\
0 & 2 & 2 & -2 & 2 \\
0 & 0 & -2 & 6 & 8 & -2
\end{bmatrix}
\]

Using back substitution, we get:

\[
e = t, d = s, c = (-\frac{1}{3})(-2 - 6s - 8t) = 1 + 3s + 4t \]

\[
b = (\frac{1}{3})(-2 - 2(1 + 3s + 4t) + 2s + 2t) = -2s - 3t, a = 2 + (-2s - 3t) - 3s - t = 2 - 5s - 4t.
\]

So, the solution is \([a, b, c, d, e] = [2 - 5s - 4t, -2s - 3t, 1 + 3s + 4t, s, t] \).
39. The key to this problem is simple substitution.

(a) The fact that \( x = t \) tells us that \( x \) is a free variable. What does that tell us?

The linear equations we are looking for must be multiples of each other. Why?

Substituting \( t = x \) into \( y = 3 - 2t \) yields \( y = 3 - 2x \Rightarrow 2x + y = 3 \).

Any multiple of this equation will create the system we are looking for.

For example, \( 2x + y = 3 \) and \( 4x + 2y = 6 \) (which is just \( 2 \times \) the equation \( 2x + y = 3 \)).

(b) Substituting \( s = y \) into \( y = 3 - 2x \) yields \( s = 3 - 2x \Rightarrow x = \frac{3}{2} - \frac{1}{2}s \).

The parametric solution then becomes \( x = \frac{3}{2} - \frac{1}{2}s \) and \( y = s \).

40. The key to this problem is simple substitution.

(a) Substituting \( t = x_1 \) into \( x_2 = 1 + t, x_3 = 2 - t \) yields \( x_2 = 1 + x_1, x_3 = 2 - x_1 \).

These equations lead immediately to the system: \( -x_1 + x_2 = 1, x_1 + x_3 = 2 \).

(b) Substituting \( s = x_3 \) into \( x_3 = 2 - x_1 \) yields \( s = 2 - x_1 \Rightarrow x_1 = 2 - s \).

Then substituting \( 2 - s = x_1 \) into \( x_2 = 1 + x_1 \) yields \( x_2 = 1 + (2 - s) \Rightarrow x_2 = 3 - s \).

The parametric solution then becomes \( x_1 = 2 - s, x_2 = 3 - s, \) and \( x_3 = s \).

41. Let \( u = \frac{1}{2}, \) and \( v = \frac{1}{4} \). Then the system of equations becomes \( 2u + 3v = 0, 3u + 4v = 1 \).

Solving the second equation for \( v \) gives \( v = \frac{3}{4} - \frac{3}{4}u \). So, substitution \( 2u + 3(\frac{3}{4} - \frac{3}{4}u) = 0 \).

Thus \( u = 3 \) and \( v = \frac{1}{4} - \frac{3}{4}(3) = -2 \). So, the solution is \( [x, y] = \left[ \frac{3}{2}, \frac{1}{2} \right] \).

42. Let \( u = x^2, \) and \( v = y^2 \). So, the system becomes \( u + 2v = 6, u - v = 3 \).

Subtracting the second equation from the first gives \( 3v = 3 \Rightarrow v = 1 \).

Substituting this into the second equation gives \( u = 3 + 1 = 4 \). Thus \( u = 4 \) and \( v = 1 \Rightarrow \) The solution set is \( [x, y] = \left[ \pm \sqrt{4}, \pm \sqrt{1} \right] \). That is, \{0, 2, -1, -1, 0, 2\}.

43. Let \( u = \tan x, \) \( v = \sin y, \) \( w = \cos z \Rightarrow u - 2v = 2, u - v + w = 2, v - w = -1 \).

We form the augmented matrix and row reduce it to find the solution of the system:

\[
\begin{bmatrix}
1 & 2 & 2 \\
1 & -1 & 1 \\
0 & 1 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & 1 \\
0 & 1 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & 1 \\
0 & 0 & -2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

Using back substitution \( w = \frac{1}{2}, v = -\frac{1}{2}, u = 2 + 2(-\frac{1}{2}) = 1 \Rightarrow [u, v, w] = \left[ 1, -\frac{1}{2}, \frac{1}{2} \right] \).

Since \( x = \tan^{-1} u, \) \( y = \sin^{-1} v, \) \( z = \cos^{-1} w, \) the solution is \( [x, y, z] = \left[ \frac{\pi}{4}, -\frac{\pi}{6}, \frac{\pi}{3} \right] \).

44. Let \( r = 2^a, \) and \( s = 3^b \). Then the system becomes \( -r + 2s = 1, 3r - 4s = 1 \).

Adding three times the first equation to the second gives \( 2s = 4 \Rightarrow s = 2 \).

Substituting \( s = 2 \) into \( -r + 2s = 1 \) yields \( -r + 2(2) = 1 \Rightarrow r = 3 \Rightarrow [r, s] = [3, 2] \).

Since \( a = \log_2 r \) and \( b = \log_3 s, \) the solution is \( [a, b] = [\log_2 3, \log_3 2] \).