HW6: 10, 14, 33, 44

Introduction to Systems of Linear Equations

1. We follow Example 2.1 and justify our assertion by applying the definition of ***linear***.
$x - \pi y + \left(\sqrt[3]{5}\right) z = 0$ ***is*** linear ***because*** power of z is 1 and π, $\sqrt[3]{5}$ are constants.

2. We follow Example 2.1 and justify our assertion by applying the definition of ***linear***.
$x^2 + y^2 + z^2 = 1$ is ***not*** linear ***because*** x, y, z occur to the power 2.

3. $x^{-1} + 7y + z = \sin \frac{\pi}{9}$ is ***not*** linear ***because*** x occurs to the power -1.

4. $2x - xy - 5z = 0$ is ***not*** linear ***because*** the product xy is of degree 2.

5. $3\cos x - 4y + z = \sqrt{3}$ is ***not*** linear ***because*** $\cos x$ is not linear.

6. $(\cos 3)\, x - 4y + z = \sqrt{3}$ ***is*** linear ***because*** $\cos 3$ and $\sqrt{3}$ are constants.

7. As in Section 1.3, we put the equation of this line into general form $ax + by = c$.
$2x + y = 7 - 3y$ is equivalent to $2x + 4y = 7$ after adding $3y$ to both sides.
Note: When the equation is ***linear*** there is no restriction on x and y. Why?

8. We begin by determining the restrictions on the variables x and y.
Typical sources are 1) division, 2) square roots, and 3) domains (like $\log x \Rightarrow x > 0$).

   Step 1. Determine restriction ***type***. With $\dfrac{x^2 - y^2}{x - y} = 1$, it is division.

   Step 2. Set the denominator equal to zero to determine the restriction.
   We have $x - y = 0 \Rightarrow x = y$. So, the ***restriction*** is $x \neq y$.

   Step 3. Simplify the given equation using algebra.
$$\frac{x^2 - y^2}{x - y} = 1 \overset{\text{factor}}{\Rightarrow} \frac{(x-y)(x+y)}{x-y} = 1 \overset{\text{cancel}}{\Rightarrow} x + y = 1.$$
Note: This tells us the given function is equivalent to the line $x + y = 1$ provided $x \neq y$.

9. We begin by determining the restrictions on the variables x and y.
Typical sources are 1) division, 2) square roots, and 3) domains (like $\log x \Rightarrow x > 0$).

   Step 1. Determine restriction ***type***. With $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{4}{xy}$, it is division.

   Step 2. Set the denominators equal to zero to determine the restriction.
   We have $x = 0$, $y = 0$, and $xy = 0$. So, the ***restriction*** is $x, y \neq 0$.

   Step 3. Simplify the given equation using algebra.
$$\frac{1}{x} + \frac{1}{y} = \frac{4}{xy} \overset{\substack{\text{common}\\ \text{denominator}}}{\Rightarrow} \frac{y}{xy} + \frac{x}{xy} = \frac{4}{xy} \overset{\substack{\text{multiply both sides}\\ \text{by xy}}}{\Rightarrow} x + y = 4.$$
Note: This tells us the given function is equivalent to the line $x + y = 4$ provided $x, y \neq 0$.

10. We begin by determining the restrictions on the variables x and y.
Typical sources are 1) division, 2) square roots, and 3) domains (like $\log x \Rightarrow x > 0$).

Step 1. Determine restriction ***type***. With $\log_{10} x - \log_{10} y = 2$, it is domains.

Step 2. Apply the domain restrictions to determine the overall restriction.
In this case, we have the overall restriction of $x > 0$ and $y > 0$.

Step 3. Simplify the given equation using algebra.

$$\log_{10} x - \log_{10} y = 2 \overset{\substack{\text{properties of}\\ \text{logarithms}}}{\Rightarrow} \log_{10} \frac{x}{y} = 2 \overset{\substack{\text{treat as}\\ \text{exponents}}}{\Rightarrow}$$

$$10^{\log_{10} \frac{x}{y}} = 10^2 \overset{\substack{\text{cancel and}\\ \text{simplify}}}{\Rightarrow} \frac{x}{y} = 100 \overset{\substack{\text{put in}\\ \text{general form}}}{\Rightarrow} x - 100y = 0.$$

Note: This tells us the given function is equivalent to the line $x - 100y = 0$ provided $x, y > 0$.

11. As in Example 2.2(a), we set $x = t$ and solve for y.
Setting $x = t$ in $3x - 6y = 0$ gives us $3t - 6y = 0$. Solving for y yields $6y = 3t \Rightarrow y = \frac{1}{2}t$.
So, we see the complete set of solutions can be written in the parametric form $[t, \frac{1}{2}t]$.

Note: We could have set $y = t$ to get $3x - 6t = 0$ and solved for x so $x = 2t$ and $[2t, t]$.

12. As in Example 2.2(a), we set $x_1 = t$ and solve for x_2.
Setting $x_1 = t$ yields $2t + 3x_2 = 5$. Solving for x_2 yields $3x_2 = 5 - 2t \Rightarrow x_2 = \frac{5}{3} - \frac{2}{3}t$.
So, a complete set of solutions written in parametric form is $[t, \frac{5}{3} - \frac{2}{3}t]$.

Note: We could have set $x_2 = t$ and solved for x_1 to get the parametric form $[\frac{5}{2} - \frac{3}{2}t, t]$.

13. As in Example 2.2(b), we set $y = s$, $z = t$ and solve for x. (Why is this a good choice?)
This substitution yields $x + 2s + 3t = 4$. Solving for x yields $x = 4 - 2s - 3t$.
So, a complete set of solutions written in parametric form is $[4 - 2s - 3t, s, t]$.

14. As in Example 2.2(b), we set $x_1 = s$, $x_2 = t$ and solve for x_3.

This substitution yields $4s + 3t + 2x_3 = 1$. Solving for x_3 yields $x_3 = \frac{1}{2} - 2s - \frac{3}{2}t$.

So, a complete set of solutions written in parametric form is $[s, t, \frac{1}{2} - 2s - \frac{3}{2}t]$.

32. The augmented matrix $\left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right]$ becomes $\begin{array}{l} a-b+\quad 3d+\ e=2 \\ a+b+2c+\ d-\ e=4 \\ b+\quad 2d+3e=0 \end{array}$.

33. As in Example 2.4(a), we add $(x-y)+(2x+y)=0+3$ to get $3x=3 \Rightarrow x=1$ and $y=1$. A quick check confirms that $[1,1]$ is indeed the unique solution of the system.

34. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 28.

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & -2 & 0 \end{array}\right] \xrightarrow[R_2+R_1]{R_2-2R_1} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/3 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/3 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \xrightarrow[R_2+R_3]{R_1-R_3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \Rightarrow$$

$x_1=\frac{2}{3}$, $x_2=-\frac{1}{3}$, and $x_3=-\frac{2}{3}$. So the solution is $[x_1,x_2,x_3]=\left[\frac{2}{3},-\frac{1}{3},-\frac{2}{3}\right]$.

35. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 29.

$$\left[\begin{array}{rr|r} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{rr|r} 1 & 5 & -1 \\ 0 & 6 & -6 \\ 0 & -6 & 6 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{rr|r} 1 & 5 & -1 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$y=-1$ and $x=-1-5(-1)=4$, so the solution is $[x,y]=[4,-1]$.

36. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 30.

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ -1 & 1 & -1 & -3 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 & 3 \end{array}\right] \Rightarrow$$

$d=t$, $c=s$, $b=-3-s-2t$, and $a=2+2(-3-s-2t)-t=-4-2s-5t$, so the solution is $[a,b,c,d]=[-4-2s-5t,-3-s-2t,s,t]$.

37. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 31.

$$\left[\begin{array}{rrr|r} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow[2R_2]{R_1 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 2 & -2 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$\Rightarrow 0=2 \Rightarrow$ No solution.

38. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 32.

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right] \xrightarrow[2R_3]{R_2-R_1} \left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & -2 & -2 & 2 \\ 0 & 2 & 0 & 4 & 6 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & -2 & -2 & 2 \\ 0 & 0 & -2 & 6 & 8 & -2 \end{array}\right]$$

Using back substitution, we get: $e=t$, $d=s$, $c=\left(-\frac{1}{2}\right)(-2-6s-8t)=1+3s+4t$.

$b=\left(\frac{1}{2}\right)(2-2(1+3s+4t)+2s+2t)=-2s-3t$, $a=2+(-2s-3t)-3s-t=2-5s-4t$.

So, the solution is $[a,b,c,d,e]=[2-5s-4t,-2s-3t,1+3s+4t,s,t]$.

39. The key to this problem is simple substitution.

(a) The fact that $x = t$ tells us that x is a free variable. What does that tell us?
The linear equations we are looking for must be multiples of each other. Why?
Substituting $t = x$ into $y = 3 - 2t$ yields $y = 3 - 2x \Rightarrow 2x + y = 3$.
Any multiple of this equation will create the system we are looking for.
For example, $2x + y = 3$ and $4x + 2y = 6$ (which is just $2\times$ the equation $2x + y = 3$).

(b) Substituting $s = y$ into $y = 3 - 2x$ yields $s = 3 - 2x \Rightarrow x = \frac{3}{2} - \frac{1}{2}s$.
The parametric solution then becomes $x = \frac{3}{2} - \frac{1}{2}s$ and $y = s$.

40. The key to this problem is simple substitution.

(a) Substituting $t = x_1$ into $x_2 = 1 + t$, $x_3 = 2 - t$ yields $x_2 = 1 + x_1$, $x_3 = 2 - x_1$.
These equations lead immediately to the system: $-x_1 + x_2 = 1$, $x_1 + x_3 = 2$.

(b) Substituting $s = x_3$ into $x_3 = 2 - x_1$ yields $s = 2 - x_1 \Rightarrow x_1 = 2 - s$.
Then substituting $2 - s = x_1$ into $x_2 = 1 + x_1$ yields $x_2 = 1 + (2 - s) \Rightarrow x_2 = 3 - s$.
The parametric solution then becomes $x_1 = 2 - s$, $x_2 = 3 - s$, and $x_3 = s$.

41. Let $u = \frac{1}{x}$, and $v = \frac{1}{y}$. Then the system of equations becomes $2u + 3v = 0$, $3u + 4v = 1$.
Solving the second equation for v gives $v = \frac{1}{4} - \frac{3}{4}u$. So, substitution $\Rightarrow 2u + 3\left(\frac{1}{4} - \frac{3}{4}u\right) = 0$.
Thus $u = 3$ and $v = \frac{1}{4} - \frac{3}{4}(3) = -2$. So, the solution is $[x, y] = \left[\frac{1}{3}, -\frac{1}{2}\right]$.

42. Let $u = x^2$, and $v = y^2$. So, the system becomes $u + 2v = 6$, $u - v = 3$.
Subtracting the second equation from the first gives $3v = 3 \Rightarrow v = 1$.
Substituting this into the second equation gives $u = 3 + 1 = 4$. Thus $u = 4$ and $v = 1 \Rightarrow$
The solution set is $[x, y] = \left[\pm\sqrt{4}, \pm\sqrt{1}\right]$. That is, $\{[2,1], [2,-1], [-2,1], [-2,-1]\}$.

43. Let $u = \tan x$, $v = \sin y$, $w = \cos z \Rightarrow u - 2v = 2$, $u - v + w = 2$, $v - w = -1$.
We form the augmented matrix and row reduce it to find the solution of the system:

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 2 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 \end{array}\right].$$

Using back substitution $w = \frac{1}{2}$, $v = -\frac{1}{2}$, $u = 2 + 2\left(-\frac{1}{2}\right) = 1 \Rightarrow [u, v, w] = \left[1, -\frac{1}{2}, \frac{1}{2}\right]$.
Since $x = \tan^{-1} u$, $y = \sin^{-1} v$, $z = \cos^{-1} w$, the solution is $[x, y, z] = \left[\frac{\pi}{4}, -\frac{\pi}{6}, \frac{\pi}{3}\right]$.

44. Let $r = 2^a$, and $s = 3^b$. Then the system becomes $-r + 2s = 1$, $3r - 4s = 1$.
Adding three times the first equation to the second gives $2s = 4 \Rightarrow s = 2$.
Substituting $s = 2$ into $-r + 2s = 1$ yields $-r + 2(2) = 1 \Rightarrow r = 3 \Rightarrow [r, s] = [3, 2]$.
Since $a = \log_2 r$ and $b = \log_3 s$, the solution is $[a, b] = [\log_2 3, \log_3 2]$.