29. Following Example 1.14, we calculate:

$\mathbf{u}\cdot\mathbf{v} = 1\cdot 5 + 2\cdot 6 + 3\cdot\ + 3\cdot 7 + 4\cdot 8 = 70,$

$\|\mathbf{u}\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$, and

$\|\mathbf{v}\| = \sqrt{5^2+6^2+7^2+8^2} = \sqrt{174}.$

Therefore, $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{70}{\sqrt{30}\sqrt{174}} = \dfrac{35}{3\sqrt{145}}$,

so $\theta = \cos^{-1}\left(\dfrac{35}{3\sqrt{145}}\right) \approx 0.2502$ radians or $14.34°$.

Note: To minimize error, we do not approximate until the last step.

Since $\dfrac{35}{3\sqrt{145}} \approx 0.9688639$ is a positive number close to 1,

we should expect θ to be close to but greater than $0°$.

30. To show $\triangle ABC$ is right, we need only show one pair of its sides meet at a right angle.

So, we let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{AC}$, then by the definition of *orthogonal* given prior to Example 1.16, we need only show $\mathbf{u}\cdot\mathbf{v}$, or $\mathbf{u}\cdot\mathbf{w}$, or $\mathbf{v}\cdot\mathbf{w} = 0$.

Following Example 1.1 of Section 1.1, we calculate the sides of $\triangle ABC$:

$\mathbf{u} = \overrightarrow{AB} = [1-(-3), 0-2] = [4,-2]$, $\mathbf{v} = \overrightarrow{BC} = [4-1, 6-0] = [3,6]$,

$\mathbf{w} = \overrightarrow{AC} = [4-(-3), 6-2] = [7,4]$, so $\mathbf{u}\cdot\mathbf{v} = 4\cdot 3 + (-2)\cdot 6 = 12 - 12 = 0 \Rightarrow$

The angle between $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$ is $90° \Rightarrow \triangle ABC$ is a right triangle.

Note: It is obvious that $\mathbf{v}$ is not orthogonal to $\mathbf{w}$. Why?

31. To show $\triangle ABC$ is right, we need only show one pair of its sides meet at a right angle.

So, we let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{AC}$, then by the definition of *orthogonal* given prior to Example 1.16, we need only show $\mathbf{u}\cdot\mathbf{v}$, or $\mathbf{u}\cdot\mathbf{w}$, or $\mathbf{v}\cdot\mathbf{w} = 0$.

Following Example 1.1 of Section 1.1, we calculate the sides of $\triangle ABC$:

$\mathbf{u} = \overrightarrow{AB} = [-3-1, 2-1, (-2)-(-1)] = [-4, 1, -1]$,

$\mathbf{v} = \overrightarrow{BC} = [2-(-3), 2-2, (-4-(-2)] = [5, 0, -2]$,

$\mathbf{w} = \overrightarrow{AC} = [2-1, 2-1, (-4)-(-1)] = [1, 1, -3]$.

Then $\mathbf{u}\cdot\mathbf{w} = (-4)\cdot 1 + 1\cdot 1 + (-1)\cdot(-3) = -4 + 1 + 3 = 0 \Rightarrow$

The angle between $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$ is $90° \Rightarrow \triangle ABC$ is a right triangle.

41. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix} 4-3 \\ -2-(-1) \\ 6-4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix} 5-3 \\ 0-(-1) \\ 2-4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(a) We compute the necessary values ...

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -3,$$

$$\mathbf{u}\cdot\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 6 \left(\|\mathbf{u}\| = \sqrt{6}\right),$$

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1 \end{bmatrix} \Rightarrow$$

$$\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{bmatrix} 5/2 \\ 1/2 \\ -1 \end{bmatrix} \Rightarrow$$

$$\|\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})\| = \sqrt{\left(\tfrac{5}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + (-1)^2}$$

$$= \tfrac{\sqrt{30}}{2}$$

... then substitute into the formula for $\mathcal{A}$:

$$\mathcal{A} = \tfrac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})\|$$

$$= \tfrac{1}{2}\left(\sqrt{6}\right)\left(\tfrac{\sqrt{30}}{2}\right) = \tfrac{3\sqrt{5}}{2}.$$

(b) We compute the necessary values ...

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -3,$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6},$$

$$\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3 \Rightarrow$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-3}{3\sqrt{6}} = -\frac{\sqrt{6}}{6} \Rightarrow$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1 - \left(\tfrac{-\sqrt{6}}{6}\right)^2} = \tfrac{\sqrt{30}}{6}$$

... then substitute into the formula for $\mathcal{A}$:

$$\mathcal{A} = \tfrac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$$

$$= \tfrac{1}{2}\left(\sqrt{6}\right)(3)\left(\tfrac{\sqrt{30}}{6}\right) = \tfrac{3\sqrt{5}}{2}.$$

If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ it does *not* follow that $\mathbf{v} = \mathbf{w}$.

An instructive counterexample is suggested by the remarks just prior to Example 1.16.
Since $\mathbf{0} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$, the zero vector is orthogonal to every vector.
So, if $\mathbf{u} = \mathbf{0}$, we know nothing about $\mathbf{v}$ and $\mathbf{w}$ except that they are vectors in $\mathbb{R}^n$.

However, we note that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ implies $\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$.
So, if $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, it *does* follow that $\mathbf{v} - \mathbf{w}$ is orthogonal to $\mathbf{u}$.

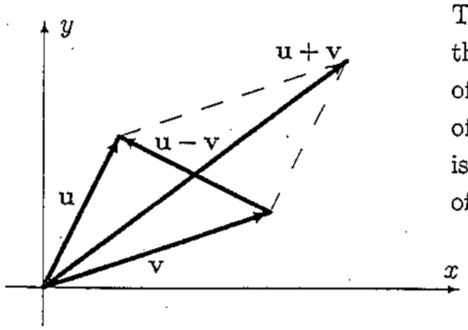
55. We need to show $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors in $\mathbb{R}^n$.
Recall, by the definitions of the dot product and the norm, $\mathbf{w} \cdot \mathbf{w} = \|\mathbf{w}\|^2$.
We apply Theorem 1.2(b) and this key fact to complete our *PROOF*:

$$\begin{aligned}(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} && \text{By Theorem 1.2(b)}\\ &= \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} && \text{By the fact that } -xy + yx = 0\\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2. && \text{By the fact that } \mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 \text{ (key fact)}\end{aligned}$$

56. (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v})\\ &= (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v})\\ &= (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) + 2\mathbf{u}\cdot\mathbf{v} + (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) - 2\mathbf{u}\cdot\mathbf{v} = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.\end{aligned}$$

(b)



The proof in part (a) tells us that the sum of the of the squares of the lengths of the diagonals of a parallelogram is twice the sum of the squares of the lengths of its sides.

57. Let $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$, and consider $\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$. By definition, we have:

$$\begin{aligned}\tfrac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \tfrac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 &= \tfrac{1}{4}[(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v})]\\ &= \tfrac{1}{4}[(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v})]\\ &= \tfrac{1}{4}[(\|\mathbf{u}\|^2 - \|\mathbf{u}\|^2) + (\|\mathbf{v}\|^2 - \|\mathbf{v}\|^2) + 4\mathbf{u}\cdot\mathbf{v}] = \mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

61. We need to show $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 2$ imply $\mathbf{u}\cdot\mathbf{v} \neq 3$.
From Theorem 1.4 (the Cauchy-Schwarz Inequality), we have $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$.
Substituting in the given values of $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 2$ shows $|\mathbf{u}\cdot\mathbf{v}| \leq 2$.
Therefore, $-2 \leq \mathbf{u}\cdot\mathbf{v} \leq 2$. It follows immediately that $\mathbf{u}\cdot\mathbf{v} \neq 3$.

62. (a) Assume that $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, so $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$.
Then $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} = 0 + 0 = 0$, so $\mathbf{u}$ is orthogonal to $\mathbf{v}+\mathbf{w}$.

(b) Assume that $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, so $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$.
Then $\mathbf{u}\cdot(s\mathbf{v}+t\mathbf{w}) = \mathbf{u}\cdot(s\mathbf{v}) + \mathbf{u}\cdot(t\mathbf{w}) = s(\mathbf{u}\cdot\mathbf{v}) + t(\mathbf{u}\cdot\mathbf{w}) = s(0) + t(0) = 0 + 0 = 0$,
so $\mathbf{u}$ is orthogonal to $s\mathbf{v}+t\mathbf{w}$.

63. Two vectors ($\mathbf{u}$ and $\mathbf{v}$) are orthogonal if their dot product equals zero. So we evaluate:

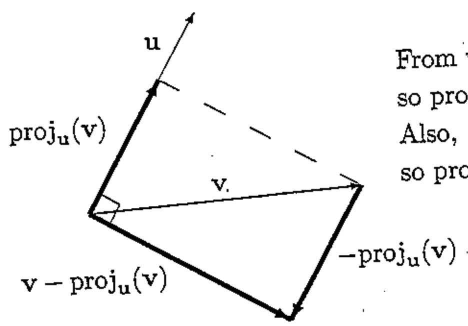
$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}-\operatorname{proj}_{\mathbf{u}}(\mathbf{v})) &= \mathbf{u}\cdot\left(\mathbf{v}-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right) = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \\ &= \mathbf{u}\cdot\mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)(\mathbf{u}\cdot\mathbf{u}) = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} = 0.\end{aligned}$$

DRAW THE TWO VECTORS. WHY DOES THIS MAKE SENSE THEY'D BE ORTHOG.?

64. (a) $\operatorname{proj}_{\mathbf{u}}(\operatorname{proj}_{\mathbf{u}}(\mathbf{v})) = \operatorname{proj}_{\mathbf{u}}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\operatorname{proj}_{\mathbf{u}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \operatorname{proj}_{\mathbf{u}}(\mathbf{v})$.

(b)
$$\begin{aligned}\operatorname{proj}_{\mathbf{u}}(\mathbf{v}-\operatorname{proj}_{\mathbf{u}}(\mathbf{v})) &= \operatorname{proj}_{\mathbf{u}}\left(\mathbf{v}-\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\left(\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} \\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = 0.\end{aligned}$$

(c)



From the diagram, we see that $\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) \parallel \mathbf{u}$,
so $\operatorname{proj}_{\mathbf{u}}(\operatorname{proj}_{\mathbf{u}}(\mathbf{v})) = \operatorname{proj}_{\mathbf{u}}(\mathbf{v})$.
Also, $(\mathbf{v}-\operatorname{proj}_{\mathbf{u}}(\mathbf{v})) \perp \mathbf{u}$,
so $\operatorname{proj}_{\mathbf{u}}(\mathbf{v}-\operatorname{proj}_{\mathbf{u}}(\mathbf{v})) = 0$.