4. (a) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(1-\lambda) - (1-\lambda) - \lambda(1-\lambda)^2$
$= -2(\lambda-1) - \lambda(\lambda-1)^2 = (\lambda-1)(\lambda-2)(\lambda+1).$

(b) $(\lambda-1)(\lambda-2)(\lambda+1) = 0 \Leftrightarrow \lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 2.$

(c) $A+I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$, so $E_{-1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right)$.

$A-I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$.

$A-2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so $E_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

(d) Each eigenvalue has algebraic and geometric multiplicity 1.

5. (a) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -(1-\lambda) + (1-\lambda)[(1-\lambda)(-1-\lambda) - (-2)]$
$= \lambda - 1 + \left(1 - \lambda + \lambda^2 - \lambda^3\right) = \lambda^2 - \lambda^3 = \lambda^2(1-\lambda).$

(b) $\lambda^2(1-\lambda) = 0 \Leftrightarrow \lambda_1 = \lambda_2 = 0,\ \lambda_3 = 1.$

(c) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $E_0 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right)$.

$A-I = \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$.

(d) 0 has algebraic multiplicity 2 and geometric multiplicity 1,
while 1 has algebraic and geometric multiplicity 1.

9. (a)
$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 4 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix}$$
$$= [(3-\lambda)(1-\lambda)+1]\left[(\lambda-1)^2-4\right] = (\lambda-2)^2(\lambda-3)(\lambda+1).$$

(b) $(\lambda-2)^2(\lambda-3)(\lambda+1)=0 \Leftrightarrow \lambda_1=-1,\ \lambda_2=\lambda_3=2,\ \lambda_4=3.$

(c)
$$A+I = \begin{bmatrix} 4 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{-1} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{2} \end{bmatrix}\right).$$
$$A-2I = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right).$$
$$A-3I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{3} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}\right).$$

(d) -1 and 3 have algebraic and geometric multiplicity 1,
while 2 has algebraic multiplicity 2 and geometric multiplicity 1.

10. (a)
$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 & 0 \\ 0 & 1-\lambda & 4 & 5 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2(3-\lambda).$$

(b) $(1-\lambda)(2-\lambda)^2(3-\lambda)=0 \Leftrightarrow \lambda_1=1,\ \lambda_2=\lambda_3=2,\ \lambda_4=3.$

(c)
$$A-I = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right).$$
$$A-2I = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 4 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{2} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right).$$
$$A-3I = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -2 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{3} = \operatorname{span}\left(\begin{bmatrix} 1 \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}\right).$$

(d) 1 and 3 have algebraic and geometric multiplicity 1,
while 2 has algebraic and geometric multiplicity 2.

13. We need to show if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x} = \lambda^{-1}\mathbf{x}$.
Since $A\mathbf{x} = \lambda\mathbf{x}$, we have $A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x}) = \lambda(A^{-1}\mathbf{x})$.
So, $\lambda(A^{-1}\mathbf{x}) = (A^{-1}A)\mathbf{x} = \mathbf{x}$ which implies $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x} = \lambda^{-1}\mathbf{x}$ as required.

14. Using induction and the proofs of 4.18(a) and (b), we will prove Theorem 4.18(c):
For any integer n if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^n\mathbf{x} = \lambda^n\mathbf{x}$.

As suggested, we will also use the fourth *Remark* following Theorem 3.9 in Section 3.3:
If A is invertible and n is a positive integer, then $A^{-n} = (A^{-1})^n = (A^n)^{-1}$.

Since (a) gives us the result for positive integers, we proceed by induction on $-n$.

1: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$.
Since $\lambda^{-1} = \frac{1}{\lambda}$, this is the statement of Theorem 4.18(b). So there is nothing to show.

k: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-k}\mathbf{x} = \lambda^{-k}\mathbf{x}$.
This is the induction hypothesis.

$k+1$: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^{-(k+1)}\mathbf{x} = \lambda^{-(k+1)}\mathbf{x}$.
This is the statement we must prove using the induction hypothesis.

$$A^{-(k+1)}\mathbf{x} \overset{\substack{\text{by}\\ \text{Remark}}}{=} A^{-1}(A^{-k}\mathbf{x}) \overset{\substack{\text{by}\\ \text{induc}}}{=} A^{-1}(\lambda^{-k}\mathbf{x}) = \lambda^{-k}(A^{-1}\mathbf{x}) \overset{\substack{\text{by}\\ n=1}}{=} \lambda^{-k}(\lambda^{-1}\mathbf{x}) = \lambda^{-(k+1)}\mathbf{x}.$$

Q: Why does the *Remark* imply that $A^{-(k+1)} = A^{-1}A^{-k}$?
A: The remark implies both $A^{-k} = (A^k)^{-1}$ and $A^{-(k+1)} = (A^{k+1})^{-1}$.
So we need only show $A^{-1}A^{-k} = (A^{k+1})^{-1}$. That is, $(A^{-1}A^{-k})(A^{k+1}) = I$.
That is obvious since: $(A^{-1}A^{-k})(A^{k+1}) = A^{-1}(A^{-k}A^k)A = A^{-1}A = I$.

Q: What does the *Remark* and our work above suggest about integer exponents of A?
A: They behave precisely as we would hope. That is, like the exponents of real variables.

15. Since $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2$ we have $A^{10}\mathbf{x} = A^{10}(2\mathbf{v}_1 + 3\mathbf{v}_2) = 2A^{10}\mathbf{v}_1 + 3A^{10}\mathbf{v}_2$.
But by Theorem 4.4(a), $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A^{10} with eigenvalues λ_1^{10} and λ_2^{10}.

So, $2\lambda_1^{10}\mathbf{v}_1 + 3\lambda_2^{10}\mathbf{v}_2 = 2\left(\frac{1}{2}\right)^{10}\mathbf{v}_1 + 3\,(2)^{10}\,\mathbf{v}_2 = \frac{1}{512}\mathbf{v}_1 + 3072\mathbf{v}_2 = \begin{bmatrix} 3072 + \frac{1}{512} \\ 3072 - \frac{1}{512} \end{bmatrix}$.

16. As in Exercise 15, $A^k\mathbf{x} = 2\left(\frac{1}{2}\right)^k\mathbf{v}_1 + 3\,(2)^k\,\mathbf{v}_2 = \begin{bmatrix} 3\cdot 2^k + 2^{1-k} \\ 3\cdot 2^k - 2^{1-k} \end{bmatrix}$.

As $k \to \infty$, the $\mathbf{v}_2$ term dominates and $A^k\mathbf{x} \approx 3\cdot 2^k\mathbf{v}_2$.

17. We must find $\mathbf{x}$ as a linear combination $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ of the eigenvectors. So:

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We must have $a_3 = 2$, so this reduces to

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

which has solution $a_1 = 1$, $a_2 = -1$. Thus,

$$A^{20}\mathbf{x} = \lambda_1^{20}\mathbf{v}_1 - \lambda_2^{20}\mathbf{v}_2 + 2 \cdot \lambda_3^{20}\mathbf{v}_3 = \begin{bmatrix} +\frac{1}{3^{20}} - \frac{1}{3^{20}} + 2 \\ -\frac{1}{3^{20}} + 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 - \frac{1}{3^{20}} \\ 2 \end{bmatrix}$$

18. As in Exercise 17, $A^k\mathbf{x} = \begin{bmatrix} 2 - \frac{2}{3^k} \\ 2 - \frac{1}{3^k} \\ 2 \end{bmatrix}$. As $k \to \infty$, $A^k\mathbf{x} \to \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

19. (a) The key observation is that $A^T - \lambda I = A^T - (\lambda I)^T = (A - \lambda I)^T$.
Thus, using Theorem 4.10, the characteristic polynomial of A^T is

$$\det\left(A^T - \lambda I\right) = \det\left(A - \lambda I\right)^T = \det\left(A - \lambda I\right)$$

But $\det(A - \lambda I)$ is the characteristic polynomial of A.
Therefore, A and A^T have the same eigenvalues as we were to show.

(b) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ has eigenspaces $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $E_2 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$,
while $A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ has eigenspaces $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

20. Given $A^n = O$, we need to show if $A\mathbf{x} = \lambda\mathbf{x}$ then $\lambda = 0$. That is:
1) 0 is an eigenvalue of A and 2) if λ is an eigenvalue of A, then $\lambda = 0$.

To prove assertion 1 we make the following observation:

Q: What is the contrapositive of Theorem 4.16?
A: A is *not* invertible if and only if 0 *is* an eigenvalue of A.

Q: What does the contrapositive of Theorem 4.16 imply?
A: $\det A = 0$ if and only if 0 *is* an eigenvalue of A. Why? Because the contrapositive of Theorem 4.6 in Section 4.2 implies $\det A = 0$ if and only if A is *not* invertible.

So, to prove 0 is an eigenvalue of A it suffices to show $\det A = 0$.
That is, if $A^n = O$, then $\det A = 0$. So, 0 is an eigenvalue of A.

Since $(\det A)^n \overset{\substack{\text{Thm 4.8}\\ \text{Sect 4.2}}}{=} \det(A^n) \overset{\substack{A^n=O\\ \text{given}}}{=} \det O = 0$, $\det A = 0$. So, 0 is an eigenvalue of A.

Next we show if λ is an eigenvalue of A, then $\lambda = 0$.

If $A\mathbf{x} = \lambda\mathbf{x}$, then Theorem 4.18(c) implies $A^n\mathbf{x} = \lambda^n\mathbf{x} = O\mathbf{x} = \mathbf{0}$.
Since $\mathbf{x}$ is an eigenvector, $\mathbf{x} \neq \mathbf{0}$. So $\lambda^n\mathbf{x} = \mathbf{0}$ implies $\lambda^n = 0$. Therefore, $\lambda = 0$ as claimed.

21. Suppose A is idempotent with eigenvector $\mathbf{x}$ corresponding to λ.
Then $\lambda\mathbf{x} = A\mathbf{x} = A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$.

So, we get $\lambda\mathbf{x} = \lambda^2\mathbf{x} \Rightarrow \lambda = \lambda^2$ (because $\mathbf{x} \neq \mathbf{0}$) $\Rightarrow \lambda^2 - \lambda = \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0$ or 1.

22. $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A\mathbf{v} - cI\mathbf{v} = \lambda\mathbf{v} - cI\mathbf{v} \Leftrightarrow (A - cI)\,\mathbf{v} = (\lambda - c)\,\mathbf{v}$.
So $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

23. (a) $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 5 & 0-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0 \Leftrightarrow \lambda = -2$ or 5.

$A + 2I = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}$, so $E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}\right)$, and $A - 5I = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}$.

So, $E_5 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

(b) By Theorem 4.4(b), A^{-1} has eigenvalues $-\frac{1}{2}$ and $\frac{1}{5}$ with

$E_{-1/2} = \text{span}\left(\begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}\right)$ and $E_{1/5} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

By Exercise 22, $A - 2I$ has eigenvalues -4 and 3 with

$E_{-4} = \text{span}\left(\begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}\right)$ and $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, and

$A + 2I$ has eigenvalues 0 and 7 with

$E_0 = \text{span}\left(\begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}\right)$ and $E_7 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

33. The characteristic polynomial $c_A(\lambda)$ of A is $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5$.
We verify that

$$A^2 - 4A + 5I = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^2 - 4\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} - \begin{bmatrix} 4 & -4 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = O.$$

34. The characteristic polynomial $c_A(\lambda)$ of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ -2 & 1 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -2 & -\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 - 3.$$

We verify that

$$-A^3 + A^2 - 3I = -\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}^3 + \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}^2 - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -1 & -1 \\ 3 & 3 & 0 \\ 3 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ -3 & 0 & 0 \\ -3 & -2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = O.$$

35. In Exercise 33, $c_A(\lambda) = \lambda^2 - 4\lambda + 5$, so $a = -4$ and $b = 5$.

Thus, $A^2 = -aA - bI = 4A - 5I = 4\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$.

Similarly, $A^3 = \left(a^2 - b\right)A + abI = 11A - 20I = 11\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}$.

The corresponding formula for A^4 is given by

$$\begin{aligned} A^4 &= AA^3 = A\left[\left(a^2 - b\right)A + abI\right] = \left(a^2 - b\right)A^2 + abA = \left(a^2 - b\right)\left[-aA - bI\right] + abA \\ &= \left(-a^3 + 2ab\right)A + \left(b^2 - a^2 b\right)I. \end{aligned}$$

So, $A^4 = 24A - 55I = 24\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - 55\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -31 & -24 \\ 48 & 17 \end{bmatrix}$.

36. For convenience, we write $-A^3 + aA^2 + bA + cI = O$, where $a = 1$, $b = 0$, $c = -3$:

$$\begin{aligned} A^3 &= aA^2 + bA + cI = A^2 - 3I = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}^2 - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ -3 & -3 & 0 \\ -3 & -2 & -2 \end{bmatrix}. \\ A^4 &= AA^3 = A\left(aA^2 + bA + cI\right) \\ &= a\left(aA^2 + bA + cI\right) + bA^2 + cA = \left(a^2 + b\right)A^2 + (ab + c)A + acI \\ &= A^2 - 3A - 3I = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}^2 - 3\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -2 & 1 \\ 0 & -3 & -3 \\ 3 & -5 & -2 \end{bmatrix}. \end{aligned}$$

37. $A^{-1} = -\frac{1}{b}A - \frac{a}{b}I = -\frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \frac{4}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$. Also,

$$\begin{aligned} A^{-2} &= \left(A^{-1}\right)^2 = \left(-\frac{1}{b}A - \frac{a}{b}I\right)^2 = \frac{1}{b^2}A^2 + \frac{2a}{b^2}A + \frac{a^2}{b^2}I = \frac{1}{b^2}(-aA - bI) + \frac{2a}{b^2}A + \frac{a^2}{b^2}I \\ &= \frac{a}{b^2}A + \frac{a^2 - b}{b^2}I = -\frac{4}{25}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \frac{11}{25}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{25} & \frac{4}{25} \\ -\frac{8}{25} & -\frac{1}{25} \end{bmatrix}. \end{aligned}$$

38. $-A^3 + aA^2 + bA + cI = -A\left(A^2 - aA - bI\right) = -cI \Leftrightarrow$

$$A^{-1} = \frac{1}{c}\left(A^2 - aA - bI\right) = -\frac{1}{3}\left(\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}^2 - \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{bmatrix}.$$

$$\begin{aligned} A^{-2} &= \left(A^{-1}\right)^2 = \left[\frac{1}{c}\left(A^2 - aA - bI\right)\right]^2 = \frac{1}{c^2}\left[A^4 - 2aA^3 + \left(a^2 - 2b\right)A^2 + 2abA + b^2 I\right] \\ &= \frac{1}{c^2}\left\{\left[\left(a^2 + b\right)A^2 + \left(ab + c\right)A + acI\right] - 2a\left[aA^2 + bA + cI\right] + \left(a^2 - 2b\right)A^2 + 2abA + b^2 I\right\} \\ &= \frac{1}{c^2}\left[-bA^2 + \left(ab + c\right)A + \left(b^2 - ac\right)I\right] \\ &= \frac{1}{9}\left(-3A + 3I\right) = \frac{1}{3}\left(-\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}. \end{aligned}$$

39. According to Exercise 69 in Section 4.2, $\det A = \det\begin{bmatrix} P & Q \\ O & S \end{bmatrix} = (\det P)(\det S)$.

We apply this result to $A - \lambda I$ instead of A below.

Note $c_A(\lambda) = \det(A - \lambda I)$, $c_P(\lambda) = \det(P - \lambda I)$, and $c_S(\lambda) = \det(S - \lambda I)$.

So, $c_A(\lambda) = \det(A - \lambda I) = \det(P - \lambda I)\det(S - \lambda I) = c_P(\lambda)c_S(\lambda)$.

Q: Why can we apply the result of Exercise 69 to $A - \lambda I$?
A: Because P becomes $P - \lambda I$ and S becomes $S - \lambda I$ so the proof holds.

That is, we have $A - \lambda I = \begin{bmatrix} P - \lambda I & Q \\ O & S - \lambda I \end{bmatrix}$.

Q: Neither O nor Q are affected. Why not?
A: Because P and S are square and I has all zero entries off the diagonal.