4.2 Determinants

1. As in Example 4.8, we compute det A by expanding along the first *row* and the first *column*.

$$row: \begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 5 & 1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = 1(1) + 3(5) = 16$$

$$column: \begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 5\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 1(1) - 5(-3) = 16$$

2. As in Example 4.8, we compute det A by expanding along the first *row* and the first *column*.

$$row: \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix} = -1\begin{vmatrix} 2 & -2 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} = -1(-2) - 1(9) = -7$$

$$column: \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -2(3) - 1(1) = -7$$

3. As in Example 4.8, we compute det A by expanding along the first *row* and the first *column*.

$$row: \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 1(-1) + 1(1) = 0$$

$$column: \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} = 1(-1) + 1(1) = 0$$

4. As in Example 4.8, we compute det A by expanding along the first *row* and the first *column*.

$$row: \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1(-1) - 1(1) = -2$$

$$column: \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1(-1) - 1(1) = -2$$

5. As in Example 4.8, we compute det A by expanding along the first *row* and the first *column*.

$$row: \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 1(5) - 2(1) + 3(-7) = -18$$

$$column: \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 1(5) - 2(1) + 3(-7) = -18$$

6. As in Example 4.8, we compute $\det A$ by expanding along the first *row* and the first *column*.

$$\textit{row}: \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 3(-1) - 2(-6) + 3(-3) = 0$$

$$\textit{column}: \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 3(-1) - 4(-6) + 7(-3) = 0$$

7. As in Example 4.10, we choose a row or column that minimizes the number of calculations. Since $\mathbf{A}_3 = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$ contains two zeroes, $\det A = a_{31}C_{31} = a_{31}(-1)^{3+1} \det A_{31} = 3 \det A_{31}$.

$$\textit{row } 3: \quad \begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 3(2) = 6$$

Q: What should we look for when choosing a row or column to expand along?
A: A row or column with the maximum number of zeroes. Why?
The maximum number of zeroes minimizes the number of cofactors we have to compute.

8. As in Example 4.10, we choose a row or column that minimizes the number of calculations. Since $\mathbf{A}_2 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$ contains one zero, $\det A = -2 \det A_{21} - 1 \det A_{23}$.

$$\textit{row } 2: \quad \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2(-1) - 1(-5) = 7$$

Q: Why is the coefficient of $\det A_{21}$ equal to -2 instead of 2?
A: Because the cofactor $C_{21} = (-1)^{2+1} \det A_{21} = -\det A_{21}$.

Q: Why is the coefficient of $\det A_{23}$ equal to -1 instead of 1?

9. As in Example 4.10, we choose a row or column that minimizes the number of calculations. Since $\mathbf{A}_3 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$ contains one zero, $\det A = 1 \det A_{31} - (-1) \det A_{32}$.

$$\textit{row 3:} \quad \begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} - (-1) \begin{vmatrix} -4 & 3 \\ 2 & 4 \end{vmatrix} = 1(10) + 1(-22) = -12$$

Q: Since both row 3 and column 3 contain one zero, what makes row 3 a better choice?
A: Because the nonzero entries of row 3 are 1 and -1, but in column 3 they are 3 and 4.

10. As in Example 4.10, we choose a row or column that minimizes the number of calculations. Since $\mathbf{a}_1 = \begin{bmatrix} \cos\theta \\ 0 \\ 0 \end{bmatrix}$ contains two zeroes, $\det A = \cos\theta \det A_{11}$.

$$\textit{col 1:} \quad \begin{vmatrix} \cos\theta & \sin\theta & \tan\theta \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix} = \cos\theta \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta(\cos^2\theta + \sin^2\theta) = \cos\theta$$

14. Since $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ contains three zeroes, $\det A = -1(-1)\det A_{32} = \det A_{32}$.

If we let $B = A_{32}$ and expand along $\mathbf{B}_1$, $\det A = \det B = 2\det B_{11} - 3\det B_{12} + (-1)\det B_{13}$.

$$\textit{col } 2: \begin{vmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix} = -(-1)\begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 8.$$

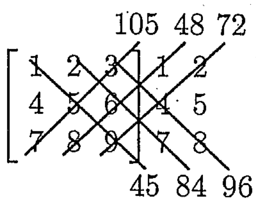
Q: What does $\det A = 8$ tell us about the rows and columns of A and B?

15. Since $\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & a \end{bmatrix}$ contains three zeroes, $\det A = -a\det A_{14}$.
If we let $B = A_{14}$ and expand along $\mathbf{B}_1$, $\det A = -a\det B = -a(b\det B_{13})$.

$$\textit{row } 1: \begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{vmatrix} = -a\begin{vmatrix} 0 & 0 & b \\ 0 & d & e \\ g & h & i \end{vmatrix} = -ab\begin{vmatrix} 0 & d \\ g & h \end{vmatrix} = abdg.$$

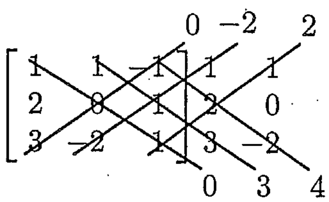
Q: Since $\det A = abdg$, if a, b, d, g are nonzero then A is invertible. Why is this obvious?
A: Since if a, b, d, g are nonzero then A is *anti*-lower triangular. Why is that enough?

16. Following the method of Example 4.9, we have:



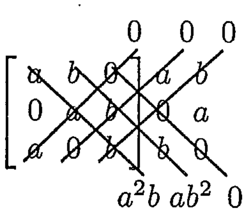
Adding the three products at the bottom and subtracting the three products at the top gives $\det A = 45 + 84 + 96 - 105 - 48 - 72 = 0$.

17. Following the method of Example 4.9, we have:



Adding the three products at the bottom and subtracting the three products at the top gives $\det A = 0 + 3 + 4 - 0 - (-2) - 2 = 7$.

18. Following the method of Example 4.9, we have:



Adding the three products at the bottom and subtracting the three products at the top gives $\det A = a^2b + ab^2 + 0 - 0 = ab(a + b)$.

25. As in Example 4.13, we use Theorem 4.3 to track adjustments to $\det A$ required $A \longrightarrow U$. We use a combination of row and column operations below to gain experience in doing so.

$$\begin{vmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix} \overset{\substack{C_3 \leftrightarrow C_4 \\ C_3 \leftrightarrow C_2 \\ C_1 \leftrightarrow C_2}}{=} - \begin{vmatrix} -1 & 2 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 4 & 0 & -1 & 1 \\ -3 & 2 & 0 & 1 \end{vmatrix} \overset{\substack{C_2+2C_1 \\ C_4+3C_1+13C_3}}{=} - \begin{vmatrix} -1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 8 \\ 4 & 8 & -1 & 0 \\ -3 & -4 & 0 & -8 \end{vmatrix} \overset{\substack{-R_1 \\ R_2+R_4}}{=} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 8 & -1 & 0 \\ 3 & -4 & 0 & -8 \end{vmatrix} = 8$$

Q: Strictly speaking, we did not reduce A to row echelon form. Why is this still sufficient?
A: The resulting matrix L is lower triangular, so $\det L$ is the product of the diagonal entries.

Q: Why did we have to introduce a negative sign after step one?
A: Because we performed 3 column interchanges. Is this true for any odd number?

Q: Why did we remove the negative sign after step three?
A: Since we multiplied row 1 by -1, we had to multiply the determinant by -1.
Which part of Theorem 4.3 tells us we have to do this multiplication? Part d.

26. Since $\mathbf{A}_3 = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = 2\mathbf{A}_1$, we have $\det A = 0$.

Q: Does Theorem 4.3 imply if $\mathbf{A}_i = k\mathbf{A}_j$ then $\det A = 0$?
A: Yes. How? Part f. asserts if $A \overset{R_i - kR_j}{\longrightarrow} B$, then $\det A = \det B$.
But $\mathbf{B}_i$ (row i of matrix B) is zero, so part c. says $\det B = \det A = 0$.

Q: Does the same hold true for columns? That is, does $\mathbf{a}_i = k\mathbf{a}_j$ imply $\det A = 0$?
A: Yes. Since $\det A = \det A^T$ this conclusion holds for both rows and columns.

Q: How might we word this conclusion as part g. (so to speak) of Theorem 4.3?

Q: These two statements are specific cases of what Theorem from this section?
A: Theorem 4.6 which asserts A is invertible if and only if $\det A \neq 0$.

Q: So what does Theorem 4.6 imply if A is *not* invertible?
A: Theorem 4.6 implies A is not invertible if and only if $\det A = 0$.

Q: How is this a generalization of the statements we have just proven?
A: We have shown if one row of A is a multiple of another row, then $\det A = 0$.
But we know that if one row of A is a multiple of another row, then A is not invertible.
Theorem 4.6 implies if there is *any* dependence relation among the rows then $\det A = 0$.
Why? Because the existence of any such dependence relation implies A is not invertible.

Q: Does Theorem 4.6 imply $\det A = 0$ if there is a dependence relation among the columns?
A: Yes. Since $\det A = \det A^T$ this conclusion holds for both rows and columns.

27. Since A is triangular, we have $\det A = a_{11}a_{22}a_{33} = (3)(-2)(4) = -24$.

Q: Which Theorem from this Section did we have employ in reaching this conclusion?
A: Theorem 4.2.

Q: How does the proof of this Theorem in Exercise 21 suggest solving this problem directly?
A: By expanding along row 3. So:

$$\text{row 3:} \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 4\begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} = 4(3)(-2) = -24$$

Note: We should apply our proofs to specific examples to see if they make sense and work. Also: A_{33} is 2×2 and upper triangular as it should be according to proof in Exercise 21.

Q: Since $A \longrightarrow I$ (obviously), why is $\det A = -24 \neq 1 = \det I$?
A: For $A \longrightarrow I$ we have to multiply row 1 by $\frac{1}{3}$, row 2 by $-\frac{1}{2}$, and row 3 by $\frac{1}{4}$. What does that tell us? We have to do the same thing to $\det A$. Therefore $\det I = \left(\frac{1}{3}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)\det A = -\frac{1}{24}(-24) = 1$.

28. Since $A \overset{R_1 \leftrightarrow R_3}{\longrightarrow} B$, $\det A = -\det B = -(3)(5)(1) = -15$ because B is triangular.

Q: Why did we have to introduce a negative sign?
A: Because we performed 1 row interchange. Is this true for any odd number? Yes.

Q: If we call this type of matrix *anti*-triangular, is the following statement true:
If A is anti-triangular, then $\pm\det A$ equals the product of the entries on the anti-diagonal.
Hint: What are the only type of operations required to make A triangular?

Q: If A is $n \times n$ and *anti*-triangular, is the following statement true:
$(-1)^n \det A$ equals the product of the entries on the anti-diagonal.
If n is even, do we have to make an even number of row interchanges to make A triangular?

29. Since $\mathbf{a}_3 = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = -2\mathbf{a}_1$, we have $\det A = 0$.

Q: Does Theorem 4.3 imply if $\mathbf{a}_i = k\mathbf{a}_j$ then $\det A = 0$?
A: Yes. Thm 4.6 implies if there is *any* dependence relation among the columns, $\det A = 0$. For a full discussion see questions and answers following Exercise 26 above.

30. Since $\mathbf{A}_3 = \begin{bmatrix} 1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 1 \end{bmatrix} = \mathbf{A}_1 + \mathbf{A}_2$, $\det A = 0$.

Q: Does Theorem 4.3 imply if $\mathbf{A}_i = c_j\mathbf{A}_j + c_k\mathbf{A}_k$ then $\det A = 0$?
A: Yes. Thm 4.6 implies if there is *any* dependence relation among the rows then $\det A = 0$. For a full discussion see questions and answers following Exercise 26 above.

31. Since $\mathbf{a}_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \mathbf{a}_1 - \mathbf{a}_2$, $\det A = 0$.

Q: Does Theorem 4.3 imply if $\mathbf{a}_i = c_j\mathbf{a}_j + c_k\mathbf{a}_k$ then $\det A = 0$?
A: Yes. Thm 4.6 implies if there is *any* dependence relation among the columns, $\det A = 0$. For a full discussion see questions and answers following Exercise 26 above.

32. Since $A \xrightarrow{R_2 \leftrightarrow R_3} B$, $\det A = -\det B = -(1)(1)(1)(1) = -1$ because B is triangular.

Q: Let E_{ij} be the elementary matrix interchanging row i and row j. Is $\det E_{ij} = -1$?
A: Yes. This is precisely the assertion of Theorem 4.4, part a.

Q: Can we generalize this result?
A: Hint: *Some* of the off diagonal entries do not affect our conclusion here.

33. Since $A \xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_2} B$, $\det A = \det B = (-3)(2)(1)(4) = -24$ because B is triangular.

Q: In $A \longrightarrow B$ we used row interchanges, but $\det A = \det B$. Why?
A: Because we performed 2 row interchanges. Is this true for any even number? Yes.

34. Since $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{A}_3 + \mathbf{A}_4$, we have $\det A = 0$.

Q: What Theorem from this section supports our conclusion above that $\det A = 0$?
A: Theorem 4.6. It implies if there is *any* dependence relation among the rows then $\det A = 0$. For a full discussion see questions and answers following Exercise 26 above.

35. Let A be the matrix given at the beginning of Exercises 35 – 40 with $\det A = 4$. Let B be the matrix given in this Exercise which is derived from A. So:
Since $A \xrightarrow{2R_1} B$, $\det B = 2 \det A = 2(4) = 8$.

Q: What Theorem from this section supports our conclusion above that $\det B = 2 \det A$?
A: Theorem 4.3 part d. which asserts if $A \xrightarrow{kR_i} B$, then $\det B = k \det A$.

36. Let A be the matrix given at the beginning of Exercises 35 – 40 with $\det A = 4$. Let D be the matrix given in this Exercise which is derived from A in 3 steps.
$A \xrightarrow{3C_1} B \xrightarrow{-C_2} C \xrightarrow{2C_3} D$, so $\det D = 2 \det C = 2(-\det B) = 2(-(3 \det A)) = -6 \det A = -24$.

Q: How might we generalize this result?
A: If $A \xrightarrow{bC_i} B \xrightarrow{cC_j} C \xrightarrow{dC_k} D$, $\det D = bcd(\det A)$.

Q: Does this result hold for rows? If $A \xrightarrow{bR_i} B \xrightarrow{cR_j} C \xrightarrow{dR_k} D$, does $\det D = bcd(\det A)$?
A: Yes. Since $\det A = \det A^T$ this result holds for both rows and columns.

45. Theorem 4.6 asserts A is invertible if and only if $\det A \neq 0$.
So, the contrapositive of Theorem 4.6 is: A is *not* invertible if and only if $\det A = 0$.
So we solve $\det A = 0$ to find the values of k we need to exclude in order for A to be invertible.

$$\det A = \begin{vmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{vmatrix} \overset{R_3-R_1}{=} \begin{vmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ 0 & k-8 & k-4 \end{vmatrix} \overset{\substack{\text{along} \\ C_1}}{=} k \begin{vmatrix} k+1 & 1 \\ k-8 & k-4 \end{vmatrix}$$

$$= k\left(k^2 - 4k + 4\right) = k\left(k-2\right)^2 = 0 \Rightarrow k = 0, 2.$$

So, the values of k we need to exclude in order for A to be invertible are $k = 0, 2$.
That is, A is invertible if and only if $k \neq 0, 2$.

46. Since Theorem 4.6 asserts A is invertible if and only if $\det A \neq 0$,
we solve $\det A = 0$ to find the values of k we need to exclude in order for A to be invertible.

$$\det A = \begin{vmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{vmatrix} \overset{C_2-C_1-C_3}{=} \begin{vmatrix} k & 0 & 0 \\ k^2 & 2-k^2-k & k \\ 0 & 0 & k \end{vmatrix} \overset{\substack{\text{along} \\ R_1}}{=} k \begin{vmatrix} 2-k^2-k & k \\ 0 & k \end{vmatrix}$$

$$= k(2-k^2-k)k = -k^2(k+2)(k-1) = 0 \Rightarrow k = -2, 0, 1.$$

So, the values of k we need to exclude in order for A to be invertible are $k = -2, 0, 1$.
That is, A is invertible if and only if $k \neq -2, 0, 1$.

47. We use Theorem 4.8, $\det(AB) = (\det A)(\det B)$, and the given values to compute $\det(AB)$.

$$\det(AB) \overset{\text{Thm 4.8}}{=} (\det A)(\det B) \overset{\text{givens}}{=} (3)(-2) = -6$$

Q: How might we state the conclusion $\det(AB) = (\det A)(\det B)$ in words?
A: The determinant of the product equals the product of the determinant.

48. We use Theorem 4.8 and $\det A = 3$ to compute $\det(B^{-1}A)$.

$$\det(A^2) = \det(A \cdot A) \overset{\text{Thm 4.8}}{=} (\det A)(\det A) = (\det A)^2 = \overset{\text{givens}}{=} (3)^2 = 9$$

Q: How might we state the conclusion $\det(A^2) = (\det A)^2$ in words?
A: The determinant of the square equals the square of the determinant.

49. We use Theorems 4.8, 4.9 and $\det A = 3$, $\det B = -2$ to compute $\det(B^{-1}A)$.

$$\det(B^{-1}A) \overset{\text{Thm 4.8}}{=} (\det(B^{-1}))(\det A) \overset{\text{Thm 4.9}}{=} \left(\tfrac{1}{\det B}\right)(\det A) \overset{\text{givens}}{=} (-\tfrac{1}{2})(3) = -\tfrac{3}{2}$$

50. We use Theorem 4.7 with $k = 2$ and $\det A = 3$, $\det B = -2$ to compute $\det(2A)$.

$$\det(2A) \overset{\substack{k=2 \\ \text{Thm 4.7}}}{=} 2^n \det A \overset{\text{givens}}{=} 2^n(3) = 3 \cdot 2^n$$

51. We use Theorem 4.7 with $k = 3$, Theorem 4.10, and the givens to compute $\det(3B^T)$.

$$\det(3B^T) \overset{\substack{k=3 \\ \text{Thm 4.7}}}{=} 3^n \det(B^T) \overset{\text{Thm 4.10}}{=} 3^n \det B \overset{\text{givens}}{=} 3^n(-2) = -2 \cdot 3^n$$

52. We use Theorems 4.8, 4.10 and $\det A = 3$, $\det B = -2$ to compute $\det(AA^T)$.

$$\det(AA^T) \overset{\text{Thm 4.8}}{=} (\det A)(\det A^T) \overset{\text{Thm 4.10}}{=} (\det A)(\det A) \overset{\text{givens}}{=} (3)(3) = 9$$

53. We use Theorem 4.8 to prove $\det(AB) = \det(BA)$.

$$\det(AB) \overset{\text{Thm 4.8}}{=} (\det A)(\det B) \overset{\det M \text{ is a scalar}}{=} (\det B)(\det A) \overset{\text{Thm 4.8}}{=} \det(BA)$$

Q: What is the key insight to take away from this exercise?
A: That $\det A$ is a scalar (not a matrix), so it commutes.

54. We use Exercise 53, associativity, and $MM^{-1} = I$ to prove $\det(B^{-1}AB) = \det A$.

$$\det(B^{-1}AB) \overset{\text{assoc}}{=} \det(B^{-1}(AB)) \overset{\text{Ex 53}}{=} \det((AB)B^{-1}) \overset{\text{assoc}}{=} \det(A(BB^{-1})) \overset{MM^{-1}=I}{=} \det A$$

Q: What is the key insight to take away from this exercise?
A: That it is important to justify each step and pay attention to the details.

Q: Could we have proven this equality directly as well? How?
A: Use Thms 4.8, 4.10, and $\left(\frac{1}{k}\right) k = 1$ to prove $\det(B^{-1}AB) = \det A$.

$$\det(B^{-1}AB) \overset{\text{Thm 4.8}}{=} (\det(B^{-1}))(\det A)(\det B) \overset{\text{Thm 4.10}}{=} \left(\frac{1}{\det B}\right)(\det A)(\det B) \overset{(1/k)k=1}{=} \det A$$

Q: What detail do we need to prove to complete the proof just given?
A: We need to show $\det(ABC) = (\det A)(\det B)(\det C)$.

Q: What are the strengths and weaknesses of the two proofs given?

Q: Could we construct a proof based on the elementary matrices?
A: See Section 3.3 where B and B^{-1} are constructed from elementary matrices.

55. We use the fact that $A^2 = A \overset{\text{Ex 47}}{\Rightarrow} (\det A)^2 = \det A$ to find all possible values of $\det A$.

$$(\det A)^2 = \det A \Rightarrow (\det A)^2 - \det A = 0 \Rightarrow \det A(\det A - 1) = 0 \Rightarrow \det A = 0, 1$$

Q: If we let $x = \det A$, what does the above calculation look like?
A: With $x = \det A$: $x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$.
This clarifies the algebra and helps us remember that $\det A$ is a scalar.

56. First we use induction on Theorem 4.8 to show that $\det(A^m) = (\det A)^m$.
Then we use that fact to find all possible values of $\det A$ when $A^m = O$.

1: $\det(A^1) = (\det A)^1$
This is obvious, so there is nothing to show.

k: $\det(A^k) = (\det A)^k$
This is the induction hypothesis, so there is nothing to show.

$k+1$: $\det(A^{k+1}) = (\det A)^{k+1}$
This is the statement we must prove using the induction on Theorem 4.8.

$$\det(A^{k+1}) = \det(AA^k) \overset{\text{Thm 4.8}}{=} \det A \det(A^k) \overset{\text{induction}}{=} \det A(\det A)^k = (\det A)^{k+1}$$

Therefore, $\det(A^m) = (\det A)^m$ as we were to show.

Since $(\det A)^m = \det(A^m) \overset{\text{given}}{=} \det O \overset{\text{Thm 4.3(a)}}{=} 0$, the only possible value for $\det A$ is zero.