1.2 Length and Angle: The Dot Product

1. Following Example 1.8, $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-1)\cdot 3 + 2\cdot 1 = -3+2 = -1.$

2. Following Example 1.8, $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ 6 \end{bmatrix} = 3\cdot 4 + (-2)\cdot 6 = 12-12 = 0.$

3. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 1\cdot 2 + 2\cdot 3 + 3\cdot 1 = 2+6+3 = 11.$

4. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 3.2 \\ -0.6 \\ -1.4 \end{bmatrix}\cdot\begin{bmatrix} 1.5 \\ 4.1 \\ -0.2 \end{bmatrix} = (3.2)\cdot(1.5) + (-0.6)\cdot(4.1) + (-1.4)\cdot(-0.2) = 2.62.$

5. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ -\sqrt{2} \\ 0 \\ -5 \end{bmatrix} = 1\cdot 4 + (\sqrt{2})\cdot(-\sqrt{2}) + \sqrt{3}\cdot 0 + 0\cdot(-5) = 4-2 = 2.$

6. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1.12 \\ -3.25 \\ 2.07 \\ -1.83 \end{bmatrix}\cdot\begin{bmatrix} -2.29 \\ 1.72 \\ 4.33 \\ -1.54 \end{bmatrix} = -1.12\cdot 2.29 + 1.72\cdot 4.33 + 1.83\cdot 1.54 = 3.6265.$

7. In the remarks prior to Example 1.11, we note that finding a unit vector $\mathbf{v}$ in the same direction as a given vector $\mathbf{u}$ is called *normalizing* the vector $\mathbf{u}$.

   Therefore, we proceed as in Example 1.12:

   $\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$, so a unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

   $$\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{5})\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

8. Following Example 1.12, we have:

   $\|\mathbf{u}\| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$, so a unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

   $$\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{13})\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{13} \\ -2/\sqrt{13} \end{bmatrix}.$$

9. Following Example 1.12, we have:

   $\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so a unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

   $$\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{14})\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}.$$

10. Following Example 1.12, we have:

$\|\mathbf{u}\| = \sqrt{(3.2)^2 + (-0.6)^2 + (-1.4)^2} = \sqrt{12.56}$, so unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is $\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{12.56})\begin{bmatrix} 3.2 \\ -0.6 \\ -1.4 \end{bmatrix} = \begin{bmatrix} 3.2/\sqrt{12.56} \\ -0.6/\sqrt{12.56} \\ -1.4/\sqrt{12.56} \end{bmatrix} \approx \begin{bmatrix} 0.2548 \\ -0.0478 \\ -0.1115 \end{bmatrix}$.

11. $\|\mathbf{u}\| = \sqrt{1^2 + (\sqrt{2})^2 + (\sqrt{3})^2 + 0^2} = \sqrt{6}$, so a unit vector in the direction of $\mathbf{u}$ is

$$\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{6})\begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2}/\sqrt{6} \\ \sqrt{3}/\sqrt{6} \\ 0/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{3}/3 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}.$$

12. $\|\mathbf{u}\| = \sqrt{(1.12)^2 + (-3.25)^2 + (2.07)^2 + (-1.83)^2} = \sqrt{19.4507}$, so the unit vector $\mathbf{v}$ is

$$\mathbf{v} = (1/\|\mathbf{u}\|)\,\mathbf{u} = (1/\sqrt{19.4507})\begin{bmatrix} 1.12 \\ -3.25 \\ 2.07 \\ -1.83 \end{bmatrix} = \begin{bmatrix} 1.12/\sqrt{19.4507} \\ -3.25/\sqrt{19.4507} \\ 2.07/\sqrt{19.4507} \\ -1.83/\sqrt{19.4507} \end{bmatrix} \approx \begin{bmatrix} 0.2540 \\ -0.7369 \\ 0.4694 \\ -0.4149 \end{bmatrix}.$$

13. Following Example 1.13, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}.$$

14. Following Example 1.13, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-8)^2} = \sqrt{65}.$$

15. Following Example 1.13, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

16. Following Example 1.13, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 3.2 \\ -0.6 \\ -1.4 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 4.1 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 1.7 \\ -4.7 \\ -1.2 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(1.7)^2 + (-4.7)^2 + (-1.2)^2} = \sqrt{26.42} \approx 5.14.$$

25. As in Example 1.14, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ (because if $\mathbf{u}\cdot\mathbf{v}=0$ we're done. Why?):

$\mathbf{u}\cdot\mathbf{v} = 2\cdot 1 + (-1)\cdot(-2) + 1\cdot(-1) = 2+2-1 = 3,$

$\|\mathbf{u}\| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$, and $\|\mathbf{v}\| = \sqrt{1^2+(-2)^2+(-1)^2} = \sqrt{6}.$

Therefore, $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{3}{\sqrt{6}\sqrt{6}} = \dfrac{1}{2}$, so $\theta = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$ radians or $60°$.

26. As in Example 1.14, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$:

$\mathbf{u}\cdot\mathbf{v} = 4\cdot 1 + 3\cdot(-1) + (-1)\cdot 1 = 4-3-1 = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta$ is right.

If we wished to be more explicit, we could continue following Example 1.15:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{0}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = 0, \text{ so } \theta = \cos^{-1}(0) = \frac{\pi}{2} \text{ radians or } 90°.$$

27. Following Example 1.14, we calculate:

$\mathbf{u}\cdot\mathbf{v} = (0.9)\cdot(-4.5) + (2.1)\cdot(2.6) + (1.2)\cdot(-0.8) = 0.45,$

$\|\mathbf{u}\| = \sqrt{(0.9)^2+(2.1)^2+(1.2)^2} = \sqrt{6.66}$, and

$\|\mathbf{v}\| = \sqrt{(-4.5)^2+(2.6)^2+(-0.8)^2} = \sqrt{27.65}.$

Therefore, $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{0.45}{\sqrt{6.66}\sqrt{27.65}} = \dfrac{0.45}{\sqrt{182.817}}$,

so $\theta = \cos^{-1}\left(\dfrac{0.45}{\sqrt{182.817}}\right) \approx 1.5375$ radians or $88.09°$.

Note: To minimize error, we do not approximate until the last step.

Since $\dfrac{0.45}{\sqrt{182.817}} \approx 0.0332816$ is a positive number close to zero,

we should expect θ to be close to but less than $90°$. Why?

28. Following Example 1.14, we calculate:

$\mathbf{u}\cdot\mathbf{v} = 1\cdot(-3) + (-2)\cdot 1 + 3\cdot(-1) + 3\cdot 4 = -4,$

$\|\mathbf{u}\| = \sqrt{1^2+(-2)^2+3^2+4^2} = \sqrt{30}$, and

$\|\mathbf{v}\| = \sqrt{(-3)^2+1^2+(-1)^2+1^2} = \sqrt{12}.$

Therefore, $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{-4}{\sqrt{30}\sqrt{12}} = -\dfrac{2}{3\sqrt{10}}$,

so $\theta = \cos^{-1}\left(-\dfrac{2}{3\sqrt{10}}\right) \approx 1.7832$ radians or $102.17°$.

Note: To minimize error, we do not approximate until the last step.

Since $-\dfrac{2}{3\sqrt{10}} \approx -0.2108185$ is a negative number close to zero,

we should expect θ to be close to but greater than $90°$. Why?

PB
17. (a) $\mathbf{u}\cdot\mathbf{v}$ is a real number, so $\|\mathbf{u}\cdot\mathbf{v}\|$ is the norm of a number, which is not defined.
(b) $\mathbf{u}\cdot\mathbf{v}$ is a scalar, while $\mathbf{w}$ is a vector.
Thus, $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ adds a scalar to a vector, which is not a defined operation.
(c) $\mathbf{u}$ is a vector, while $\mathbf{v}\cdot\mathbf{w}$ is a scalar.
Thus, $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ is the dot product of a vector and a scalar, which is not defined.
(d) $c\cdot(\mathbf{u}+\mathbf{v})$ is the dot product of a scalar and a vector, which is not defined.

18. From trigonometry, we have:
$\cos\theta>0\Rightarrow\theta$ is acute, $\cos\theta<0\Rightarrow\theta$ is obtuse, and $\cos\theta=0\Rightarrow\theta$ is right.
From $\cos\theta=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$, we see $\mathbf{u}\cdot\mathbf{v}$ determines the sign of $\cos\theta$. Why?
Therefore, as in Example 1.14, we calculate:
$\mathbf{u}\cdot\mathbf{v}=3\cdot(-1)+0\cdot1=-3<0\Rightarrow\cos\theta<0\Rightarrow\theta$ is obtuse.

19. From trigonometry, we have:
$\cos\theta>0\Rightarrow\theta$ is acute, $\cos\theta<0\Rightarrow\theta$ is obtuse, and $\cos\theta=0\Rightarrow\theta$ is right.
From $\cos\theta=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$, we see $\mathbf{u}\cdot\mathbf{v}$ determines the sign of $\cos\theta$. Why?
Therefore, as in Example 1.14, we calculate:
$\mathbf{u}\cdot\mathbf{v}=2\cdot1+(-1)\cdot(-2)+1\cdot(-1)=4>0\Rightarrow\cos\theta>0\Rightarrow\theta$ is acute.

20. Following the first step in Example 1.14, we calculate:
$\mathbf{u}\cdot\mathbf{v}=4\cdot1+3\cdot(-1)+(-1)\cdot1=4-3-1=0\Rightarrow\cos\theta=0\Rightarrow\theta$ is right.

21. Following the first step in Example 1.14, we calculate:
$\mathbf{u}\cdot\mathbf{v}=(0.9)\cdot(-4.5)+(2.1)\cdot(2.6)+(1.2)\cdot(-0.8)=0.45\Rightarrow\cos\theta>0\Rightarrow\theta$ is acute.

22. $\mathbf{u}\cdot\mathbf{v}=1\cdot(-3)+(-2)\cdot1+3\cdot(-1)+3\cdot4=-4\Rightarrow\cos\theta<0\Rightarrow\theta$ is obtuse.

23. Since $\mathbf{u}\cdot\mathbf{v}$ is obviously >0, we have $\cos\theta>0$ which implies θ is acute.
Note: $\mathbf{u}\cdot\mathbf{v}$ is >0 because the components of both $\mathbf{u}$ and $\mathbf{v}$ are positive.

24. As in Example 1.14, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ (if $\mathbf{u}\cdot\mathbf{v}=0$, we're done. Why?):
$\mathbf{u}\cdot\mathbf{v}=3\cdot(-1)+0\cdot1=-3$, $\|\mathbf{u}\|=\sqrt{3^2+0^2}=\sqrt{9}=3$, $\|\mathbf{v}\|=\sqrt{(-1)^2+1^2}=\sqrt{2}$.
So, $\cos\theta=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\dfrac{-3}{3\sqrt{2}}=-\dfrac{\sqrt{2}}{2}$ and $\theta=\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)=\dfrac{3\pi}{4}$ radians or $135°$.

42. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal ***if and only if*** [$\Leftrightarrow$] their dot product is zero. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So, we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} k+1 \\ k-1 \end{bmatrix} = 0 \Rightarrow 2(k+1) + 3(k-1) = 0 \Rightarrow 5k - 1 = 0 \Rightarrow k = \tfrac{1}{5}.$$

Substituting k back into the expression for $\mathbf{v}$ we get: $\mathbf{v} = \begin{bmatrix} \frac{1}{5}+1 \\ \frac{1}{5}-1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix}$.

We check our answer by computing $\mathbf{u} \cdot \mathbf{v}$ (it should be zero):

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix} = \frac{12}{5} - \frac{12}{5} = 0 \text{ as required.}$$

43. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal ***if and only if*** [$\Leftrightarrow$] their dot product is zero. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So, we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix} = 0 \Rightarrow k^2 - k - 6 = (k+2)(k-3) = 0 \Rightarrow k = -2,\ 3.$$

Substituting k back into the expression for $\mathbf{v}$ we get:

When $k = -2$, $\mathbf{v}_1 = \begin{bmatrix} (-2)^2 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$. When $k = 3$, $\mathbf{v}_2 = \begin{bmatrix} 3^2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix}$.

We check by computing $\mathbf{u} \cdot \mathbf{v}_1$ and $\mathbf{u} \cdot \mathbf{v}_2$ (they should both be zero):

$$\mathbf{u} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} = 4 + 2 - 6 = 0 \text{ and } \mathbf{u} \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix} = 9 - 3 - 6 = 0.$$

R)

44. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal ***if and only if*** [$\Leftrightarrow$] their dot product is zero. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So, we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for y in terms of x:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow 3x + y = 0 \Rightarrow y = -3x.$$

Substituting $y = -3x$ back into the expression for $\mathbf{v}$ we get: $\mathbf{v} = \begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Conclusion: Any vector orthogonal to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ must be a multiple of $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Check: $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ -3x \end{bmatrix} = 3x - 3x = 0$ for all values of x.

Note: We could also have solved for x in terms of y yielding $\mathbf{v} = \begin{bmatrix} -\frac{1}{3}y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$.

47. We prove Theorem 1.2(b) by applying the definition of the dot product.

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= u_1(v_1+w_1)+u_2(v_2+w_2)+\cdots+u_n(v_n+w_n)\\ &= u_1v_1+u_1w_1+u_2v_2+u_2w_2+\cdots+u_nv_n+u_nw_n\\ &= (u_1v_1+u_2v_2+\cdots+u_nw_n)+(u_1w_1+u_2w_2+\cdots+u_nw_n)\\ &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}.\end{aligned}$$

48. We prove the three parts of Theorem 1.2(d) by applying the definition of the dot product and key properties of real numbers.

Part 1: For any vector $\mathbf{u}$, we need to show $\mathbf{u}\cdot\mathbf{u}\geq 0$.
We begin by noting that for any real number x, we have $x^2\geq 0$.
So, $\mathbf{u}\cdot\mathbf{u}=u_1u_1+u_2u_2+\cdots+u_nu_n=u_1^2+u_2^2+\cdots+u_n^2\geq 0$.
Note: $u_1^2+u_2^2+\cdots+u_n^2\geq 0$ because the u_i are real numbers.

Part 2: We need to show if $\mathbf{u}=\mathbf{0}$, then $\mathbf{u}\cdot\mathbf{u}=0$.
We begin by noting that if $\mathbf{u}=\mathbf{0}$, then $u_i=0$ for all i.
If $\mathbf{u}=\mathbf{0}$, then $\mathbf{u}\cdot\mathbf{u}=\mathbf{0}\cdot\mathbf{0}=u_1^2+u_2^2+\cdots+u_n^2=0^2+0^2+\cdots+0^2=0$.

Part 3: We need to show if $\mathbf{u}\cdot\mathbf{u}=0$, then $\mathbf{u}=\mathbf{0}$.
We begin by noting that for any real number x, if $x^2=0$ then $x=0$.
If $\mathbf{u}\cdot\mathbf{u}=u_1u_1+u_2u_2+\cdots+u_nu_n=u_1^2+u_2^2+\cdots+u_n^2=0$
then $u_i^2=0$ for all i which implies $u_i=0$ because the u_i are real numbers.
Therefore, since $u_i=0$ for all i, by definition $\mathbf{u}=\mathbf{0}$.

49. We need to show $\mathrm{d}(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=\|\mathbf{v}-\mathbf{u}\|=\mathrm{d}(\mathbf{v},\mathbf{u})$.
If we let $c=-1$ in Theorem 1.3(b), then $\|-\mathbf{w}\|=\|\mathbf{w}\|$. We use this **key fact** below.

PROOF:

$\mathrm{d}(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|$	By definition
$=\|-(\mathbf{v}-\mathbf{u})\|$	By the fact that $(x-y)=-(y-x)$
$=\|\mathbf{v}-\mathbf{u}\|$	By $\|-\mathbf{w}\|=\|\mathbf{w}\|$ **(key fact)**
$=\mathrm{d}(\mathbf{v},\mathbf{u})$.	By definition

50. We need to show $\mathrm{d}(\mathbf{u},\mathbf{w})\leq\mathrm{d}(\mathbf{u},\mathbf{v})+\mathrm{d}(\mathbf{v},\mathbf{w})$. That is, $\|\mathbf{u}-\mathbf{w}\|\leq\|\mathbf{u}-\mathbf{v}\|+\|\mathbf{v}-\mathbf{w}\|$.
This follows immediately from Theorem 1.5:

$$\|\mathbf{x}+\mathbf{y}\|\leq\|\mathbf{x}\|+\|\mathbf{y}\| \quad \text{with } \mathbf{x}=\mathbf{u}-\mathbf{v} \text{ and } \mathbf{y}=\mathbf{v}-\mathbf{w}.$$

51. We need to show $\mathrm{d}(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=0$ if and only if $\mathbf{u}=\mathbf{v}$.
This follows immediately from Theorem 1.3(a): $\|\mathbf{w}\|=0$ if and only if $\mathbf{w}=\mathbf{0}$, with $\mathbf{w}=\mathbf{u}-\mathbf{v}$.

52. We will show $\mathbf{u}\cdot c\mathbf{v}=c(\mathbf{u}\cdot\mathbf{v})$ by applying the definitions.

$$\begin{aligned}\mathbf{u}\cdot c\mathbf{v} &= [u_1,u_2,\ldots,u_n]\cdot[cv_1,cv_2,\ldots,cv_n]=u_1cv_1+u_2cv_2+\cdots+u_ncv_n\\ &= cu_1v_1+cu_2v_2+\cdots+cu_nv_n=c(u_1v_1+u_2v_2+\cdots+u_nv_n)=c(\mathbf{u}\cdot\mathbf{v})\end{aligned}$$

53. We need to show $\|\mathbf{u}-\mathbf{v}\|\geq\|\mathbf{u}\|-\|\mathbf{v}\|$. That is, $\|\mathbf{u}\|\leq\|\mathbf{u}-\mathbf{v}\|+\|\mathbf{v}\|$.
This follows immediately from Theorem 1.5, $\|\mathbf{x}+\mathbf{y}\|\leq\|\mathbf{x}\|+\|\mathbf{y}\|$, with $\mathbf{x}=\mathbf{u}-\mathbf{v}$ and $\mathbf{y}=\mathbf{v}$.