4.1 Introduction to Eigenvalues and Eigenvectors

1. If \( Ax = \lambda x \), then \( x \) is an eigenvector of \( A \) corresponding to \( \lambda \).

So, as in Example 4.1, since \( Av = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3v \),
we see \( v \) is an eigenvector of \( A \) corresponding to (the eigenvalue) 3.

2. If \( Ax = \lambda x \), then \( x \) is an eigenvector of \( A \) corresponding to \( \lambda \).

So, as in Example 4.1, since \( Av = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -1 \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -1v \),
we see \( v \) is an eigenvector of \( A \) corresponding to (the eigenvalue) -1.

3. We compute \( Av = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -3v \),
we see \( v \) is an eigenvector of \( A \) corresponding to (the eigenvalue) -3.

4. We compute \( Av = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 3v \),
so \( v \) is an eigenvector of \( A \) corresponding to the eigenvalue 3.

5. We compute \( Av = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 3v \),
so \( v \) is an eigenvector of \( A \) corresponding to the eigenvalue 3.

6. We compute \( Av = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0v \),
so \( v \) is an eigenvector of \( A \) corresponding to the eigenvalue 0.
9. As in Example 4.2, we show null\((A - I) \neq 0\) then compute null\((A - I)\) to find \(x\).

Since \(Ax = x\) implies \((A - I)x = 0\), we have:

\[
A - I = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix}
\]

Since the columns of \(A - I\) are clearly linearly dependent (because \(a_2 = -4a_1\)), the Fundamental Theorem of Invertible Matrices implies that null\((A - I) \neq 0\). That is \(Ax = x\) has a nontrivial solution, so 1 is an eigenvalue of \(A\).

Since \(Ax = x\) implies \((A - I)x = 0\), we now compute null\((A - I)\).

\[
[A - I | 0] = \begin{bmatrix} -1 & 4 & 0 \\ -1 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

So, if \(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\) is an eigenvector corresponding to the eigenvalue 1, then \(x_1 = 4x_2\).

These eigenvectors are of the form \(x = \begin{bmatrix} 4x_2 \\ x_2 \end{bmatrix}\). That is nonzero multiples of \(x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}\)

Q: What does this tell us about null\((A - I)\)? What about \(E_1\)?
A: The above shows null\((A - I) = \text{span} \left( \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) = E_1\), the eigenspace of 1.

10. As in Example 4.2, we show null\((A - 4I) \neq 0\) then compute null\((A - 4I)\) to find \(x\).

Since \(Ax = 4x\) implies \((A - 4I)x = 0\), we have:

\[
A - 4I = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 0 & 1 \end{bmatrix}
\]

Since the columns of \(A - 4I\) are clearly linearly dependent (because \(a_2 = -a_1\)), the Fundamental Theorem of Invertible Matrices implies that null\((A - 4I) \neq 0\). That is \(Ax = 4x\) has a nontrivial solution, so 4 is an eigenvalue of \(A\).

Since \(Ax = 4x\) implies \((A - 4I)x = 0\), we now compute null\((A - 4I)\).

\[
[A - 4I | 0] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 0 \end{bmatrix}
\]

So, if \(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\) is an eigenvector corresponding to the eigenvalue 4, then \(x_2 = x_1\).

These eigenvectors are of the form \(x = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}\). That is nonzero multiples of \(x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\).

Q: What does this tell us about null\((A - 4I)\)? What about \(E_4\)?
A: The above shows null\((A - 4I) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = E_4\), the eigenspace of 4.
11. As in Example 4.2, we show $\text{null}(A + I) \neq 0$ then compute $\text{null}(A + I)$ to find $x$.

Since $Ax = -1x$ implies $(A + I)x = 0$, we have:

$$A + I = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

Since the columns of $A + I$ are clearly linearly dependent (because $a_3 = a_1 + a_2$), the Fundamental Theorem of Invertible Matrices implies that $\text{null}(A + I) \neq 0$. That is, $Ax = -1x$ has a nontrivial solution, so $-1$ is an eigenvalue of $A$.

Since $Ax = -1x$ implies $(A + I)x = 0$, we now compute $\text{null}(A + I)$.

$$[A + I \mid 0] = \begin{bmatrix} 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $-1$, then $x_1 = x_2 = -x_3$.

These eigenvectors are of the form $x = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix}$, nonzero multiples of $x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Q: What does this tell us about $\text{null}(A + I)$? What about $E_{-1}$?

A: The above shows $\text{null}(A + I) = \text{span} \left( \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) = E_{-1}$, the eigenspace of $-1$.

12. As in Example 4.2, we show $\text{null}(A - 2I) \neq 0$ then compute $\text{null}(A - 2I)$ to find $x$.

Since $Ax = 2x$ implies $(A - 2I)x = 0$, we have:

$$A - 2I = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

Since the columns of $A - 2I$ are clearly linearly dependent (because $a_3 = -a_2$), the Fundamental Theorem of Invertible Matrices implies that $\text{null}(A - 2I) \neq 0$. That is, $Ax = 2x$ has a nontrivial solution, so $2$ is an eigenvalue of $A$.

Since $Ax = 2x$ implies $(A - 2I)x = 0$, we now compute $\text{null}(A - 2I)$.

$$[A - 2I \mid 0] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & 2 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $2$, then $x_1 = 0$, $x_3 = x_2$.

These eigenvectors are of the form $x = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$, nonzero multiples of $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.
15. From the remarks prior to Example 4.4, we have the following key insight: 
\( \mathbf{x} \) is an eigenvector of \( A \) if and only if \( A \) transforms \( \mathbf{x} \) to a parallel vector. 
Why? Because then \( A\mathbf{x} \) and \( \mathbf{x} \) are multiples of each other. That is, \( A\mathbf{x} = \lambda \mathbf{x} \).
Recall that \( E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda \} \cup \{\text{the zero vector, } \mathbf{0} \} \).
We have to add the zero vector because eigenvectors are nonzero by definition.

Since \( A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \), \( A \) is the matrix of projection \( P \) onto the \( x \)-axis.

Consider vectors \( \mathbf{v} \) parallel to the \( x \)-axis, parallel to the \( y \)-axis, and not parallel to either axis.

- **\( x \)-axis:** If \( \mathbf{v} \) is parallel to the \( x \)-axis, \( P \) transforms \( \mathbf{v} \) to itself. That is, \( P(\mathbf{v}) = \mathbf{v} \).
  So, all nonzero vectors parallel to the \( x \)-axis are eigenvectors of \( A \) corresponding to 1.

- **\( y \)-axis:** If \( \mathbf{v} \) is parallel to the \( y \)-axis, \( P \) transforms \( \mathbf{v} \) to \( \mathbf{0} \). That is, \( P(\mathbf{v}) = \mathbf{0} \).
  So, all nonzero vectors parallel to the \( y \)-axis are eigenvectors of \( A \) corresponding to 0.

- **neither:** If \( \mathbf{v} \) is not parallel to either axis, \( P \) transforms \( \mathbf{v} \) to a nonparallel vector.
  So, all nonzero vectors not parallel to either axis are not eigenvectors of \( A \).

So \( E_1 = \text{span}(x\text{-axis}) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \) and \( E_0 = \text{span}(y\text{-axis}) = \text{span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \).

Q: Given that the \( x \)-axis is a line, how might we generalize this result?
A: Hint: Consider vectors parallel, perpendicular and neither to the given line.

16. From the remarks prior to Example 4.4, we have the following key insight: 
\( \mathbf{x} \) is an eigenvector of \( A \) if and only if \( A \) transforms \( \mathbf{x} \) to a parallel vector. 
Why? Because then \( A\mathbf{x} \) and \( \mathbf{x} \) are multiples of each other. That is, \( A\mathbf{x} = \lambda \mathbf{x} \).
Recall that \( E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda \} \cup \{\text{the zero vector, } \mathbf{0} \} \).
We have to add the zero vector because eigenvectors are nonzero by definition.

From Example 3.59 in Section 3.6, we have: 
\[
P_\ell(\mathbf{x}) = \frac{1}{d_1^2 + d_2^2} \begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

Since \( A\mathbf{x} = \frac{1}{25} \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 16x + 12y \\ 12x + 9y \end{bmatrix} = \frac{4}{25}x \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \frac{3}{25}y \begin{bmatrix} 4 \\ 3 \end{bmatrix},
\]

\( A \) is the matrix of projection \( P_\ell \) onto line \( \ell \) with direction vector \( \mathbf{d} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \).

Consider vectors \( \mathbf{v} \) parallel to \( \ell \), perpendicular to \( \ell \), and neither to direction vector \( \mathbf{d} \).

- **parallel:** If \( \mathbf{v} \) is parallel to \( \ell \), \( P_\ell \) transforms \( \mathbf{v} \) to itself. That is, \( P_\ell(\mathbf{v}) = \mathbf{v} \).
  So, all nonzero vectors parallel to \( \ell \) are eigenvectors of \( A \) corresponding to 1.

- **perpendicular:** If \( \mathbf{v} \) is perpendicular to \( \ell \), \( P_\ell \) transforms \( \mathbf{v} \) to \( \mathbf{0} \). That is, \( P_\ell(\mathbf{v}) = \mathbf{0} \).
  So, all nonzero vectors perpendicular to \( \ell \) are eigenvectors of \( A \) corresponding to 0.

- **neither:** If \( \mathbf{v} \) is neither parallel nor perpendicular, \( P_\ell \) transforms \( \mathbf{v} \) to a nonparallel vector.
  So, all vectors not parallel or perpendicular to \( \ell \) are not eigenvectors of \( A \).

\[
E_1 = \text{span}(\text{parallel to } \mathbf{d}) = \text{span} \left( \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right)
\]

\[
E_0 = \text{span}(\text{perpendicular to } \mathbf{d}) = \text{span} \left( \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right).
\]
17. From the remarks prior to Example 4.4, we have the following key insight: 
\( x \) is an eigenvector of \( A \) if and only if \( A \) transforms \( x \) to a parallel vector.  
Why? Because then \( Ax \) and \( x \) are multiples of each other. That is, \( Ax = \lambda x \).  
Recall that \( E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda \} \cup \{\text{the zero vector, } 0\} \).  
We have to add the zero vector because eigenvectors are nonzero by definition.  

Since \( Ax = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix} \), \( A \) is the matrix of stretching \( S \).  
Consider vectors \( v \) parallel to the \( x \)-axis, parallel to the \( y \)-axis, and not parallel to either axis.  

\( x \)-axis: If \( v \) is parallel to the \( x \)-axis, \( S \) transforms \( v \) to twice itself. That is, \( S(v) = 2v \).  
So, all nonzero vectors parallel to the \( x \)-axis are eigenvectors of \( A \) corresponding to 2.  
\( y \)-axis: If \( v \) is parallel to the \( y \)-axis, \( S \) transforms \( v \) to thrice itself. That is, \( S(v) = 3v \).  
So, all nonzero vectors parallel to the \( y \)-axis are eigenvectors of \( A \) corresponding to 3.  
\( \text{neither} \): If \( v \) is not parallel to either axis, \( S \) transforms \( v \) to a nonparallel vector.  
So, all vectors not parallel to either axis are not eigenvectors of \( A \).  

So \( E_2 = \text{span}(x\text{-axis}) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \) and \( E_3 = \text{span}(y\text{-axis}) = \text{span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \).  
Q: Following this exact same process, how might we generalize this result?  
A: If \( A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \), then \( E_a = \text{span}(x\text{-axis}) \) and \( E_d = \text{span}(y\text{-axis}) \).  

18. From the remarks prior to Example 4.4, we have the following key insight: 
\( x \) is an eigenvector of \( A \) if and only if \( A \) transforms \( x \) to a parallel vector.  
Why? Because then \( Ax \) and \( x \) are multiples of each other. That is, \( Ax = \lambda x \).  
Recall that \( E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda \} \cup \{\text{the zero vector, } 0\} \).  
We have to add the zero vector because eigenvectors are nonzero by definition.  

From Example 3.58 in Section 3.6, we have: \( R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \).  
\( Ax = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \), so \( A \) is the matrix of rotation \( R_{90^\circ} \).  
Consider the zero vector \( 0 \) and all nonzero vectors \( v \).  

\( v = 0 \): Since a rotation leaves the zero vector fixed, \( R_{90^\circ} (0) = 0 \).  
However, the zero vector is not an eigenvector of \( A \) corresponding to 0.  
Why not? Because the zero vector is zero and eigenvectors must be nonzero by definition.  
\( v \neq 0 \): A rotation transforms any nonzero vector to a nonparallel vector.  
So, all nonzero vectors are not eigenvectors of \( A \) when \( A \) is the matrix of any rotation.  
Q: Is it still true that \( E_0 = \text{span}(0) = 0 \)?  
A: Yes, because \( E_0 = \{\text{eigenvectors of } \lambda \} \cup \{\text{the zero vector, } 0\} = \{0\} \).
21. From the remarks prior to Example 4.4, we have the following key insight:
\( x \) is an eigenvector of \( A \) if and only if \( A \) transforms \( x \) to a parallel vector.
So, lines that do not bend at the unit circle represent eigenvectors.
The extension beyond the circle tells us if the vector has been stretched.

Since the lines do not bend on the line \( l \ y = -x \) with direction vector \( d = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \),

So, we consider vectors \( v \) parallel to \( d \), perpendicular to \( d \), and neither.

**parallel:** On the line \( y = x \), the lines do not bend and extend precisely 2 units beyond it. So:
If \( v \) is parallel to the \( d \), \( S \) transforms \( v \) to thrice itself. That is, \( S(v) = 2v \).
So, all nonzero vectors parallel to \( d \) are eigenvectors of \( A \) corresponding to 2.

**perp:** On the line \( y = -x \), the lines extend precisely 0 units beyond the unit circle. So:
If \( v \) is perpendicular to the \( d \), \( S \) transforms \( v \) to \( 0 \). That is, \( S(v) = 0v \).
So, all nonzero vectors perpendicular to \( d \) are eigenvectors of \( A \) corresponding to 0.

**neither:** Off the lines \( y = x \) and \( y = -x \), the lines do bend at the unit circle. So:
If \( v \) is not parallel or perpendicular to \( d \), \( S \) transforms \( v \) to a nonparallel vector.
So, all vectors not parallel or perpendicular to \( d \) are not eigenvectors of \( A \).

So \( E_2 = \text{span}(d) = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( E_0 = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

22. From the remarks prior to Example 4.4, we have the following key insight:
\( x \) is an eigenvector of \( A \) if and only if \( A \) transforms \( x \) to a parallel vector.
So, lines that do not bend at the unit circle represent eigenvectors.
Here, however, all lines bend at the unit circle, so we conclude there are no eigenvectors.

Q: What types of transformations have no eigenvectors?
A: Rotations. See Exercise 18. Is the graph in Exercise 22 suggestive of a rotation?
35. (a) To find the eigenvalues of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we solve $\det(A - \lambda I) = 0 \iff$

$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \text{tr}(A)\lambda + \det A = 0.$$  

(b) Using the quadratic formula, the solutions to the equation in part (a) are

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} = \frac{a + d \pm \sqrt{a^2 + d^2 + 2ad - 4ad + 4bc}}{2} = \frac{1}{2} \left(a + d \pm \sqrt{(a - d)^2 + 4bc}\right).$$

(c) Let $\lambda_1 = \frac{1}{2}(a + d) + \sqrt{(a - d)^2 + 4bc}$ and $\lambda_2 = \frac{1}{2}(a + d) - \sqrt{(a - d)^2 + 4bc})$. So, $\lambda_1 + \lambda_2 = \frac{1}{2}(a + d) + \frac{1}{2}(a + d) = a + d = \text{tr}(A)$.

Also, $\lambda_1\lambda_2 = \frac{1}{4}[(a + d)^2 - ((a - d)^2 + 4bc)] = \frac{1}{4}[4ad - 4bc] = ad - bc = \det A$.

36. (a) If $A$ is to have two distinct real eigenvalues, the discriminant of the equation in Exercise 35(b) must be positive. That is, $(a - d)^2 + 4bc > 0 \iff (a - d)^2 > -4bc$. If $a = d$, then neither $b$ nor $c$ can equal zero if this inequality is to hold.

If $a \neq d$, whenever $b$ and $c$ have the same sign this inequality holds.

(b) If $A$ is to have one real eigenvalue, the discriminant must be zero. That is, $(a - d)^2 + 4bc = 0 \iff (a - d)^2 = -4bc$.

If $a = d$, then neither $b$ or $c$ must be zero if this inequality is to hold.

(c) If $A$ is to have no real eigenvalue, the discriminant must be negative. That is, $(a - d)^2 + 4bc < 0 \iff (a - d)^2 < -4bc$.

If $a = d$, then neither $b$ nor $c$ can equal zero if this inequality is to hold.

If $a \neq d$, then $b$ and $c$ must have opposite signs if this inequality is to hold.
37. As in Example 4.5, we find all solutions $\lambda$ of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ 0 & d - \lambda \end{bmatrix} = \lambda^2 - (a + d)\lambda + ad$$

Since $\lambda^2 - (a + d)\lambda + ad = (\lambda - a)(\lambda - d) = 0$, the solutions are $\lambda = a, d$.

$\lambda = a$: $A - aI = \begin{bmatrix} a-a & b \\ 0 & a-d \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & a-d \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $a$, $x_1 = t$, $x_2 = 0$.

These eigenvectors are of the form $x = \begin{bmatrix} t \\ 0 \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So, $E_a = \text{null}(A - aI) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$.

$\lambda = d$: $A - dI = \begin{bmatrix} a - d & b \\ 0 & d - d \end{bmatrix} = \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$

If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $d$, $(d-a)x_1 = bx_2 = (d-a)bt$.

These eigenvectors are of the form $x = \begin{bmatrix} bt \\ (d-a)t \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} b \\ d-a \end{bmatrix}$.

So, $E_b = \text{null}(A - bI) = \text{span} \left( \begin{bmatrix} b \\ d-a \end{bmatrix} \right)$.

38. As in Example 4.5, we find all solutions $\lambda$ of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ -b & a - \lambda \end{bmatrix} = \lambda^2 - 2a\lambda + a^2 + b^2$$

Since $\lambda^2 - 2a\lambda + a^2 + b^2 = 0$ implies $\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$, we have $\lambda = a + bi, a - bi$.

$a + bi$: $A - (a + bi)I = \begin{bmatrix} a - (a + bi) & b \\ -b & a - (a + bi) \end{bmatrix} = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$

If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $a$, $x_2 = ix_1$.

These eigenvectors are of the form $x = \begin{bmatrix} t \\ it \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

So, $E_{a+bi} = \text{null}(A - (a + bi)I) = \text{span} \left( \begin{bmatrix} 1 \\ i \end{bmatrix} \right)$.

$a - bi$: $A - (a - bi)I = \begin{bmatrix} a - (a - bi) & b \\ -b & a - (a - bi) \end{bmatrix} = \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} \rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$

If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $a$, $x_2 = -ix_1$.

These eigenvectors are of the form $x = \begin{bmatrix} t \\ -it \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

So, $E_{a-bi} = \text{null}(A - (a - bi)I) = \text{span} \left( \begin{bmatrix} 1 \\ -i \end{bmatrix} \right)$.