4.1 Introduction to Eigenvalues and Eigenvectors

1. If $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an eigenvector of A corresponding to λ.

So, as in Example 4.1, since $A\mathbf{v} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{v}$,

we see $\mathbf{v}$ is an eigenvector of A corresponding to (the eigenvalue) 3.

2. If $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an eigenvector of A corresponding to λ.

So, as in Example 4.1, since $A\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -1\begin{bmatrix} 3 \\ -3 \end{bmatrix} = -1\mathbf{v}$,

we see $\mathbf{v}$ is an eigenvector of A corresponding to (the eigenvalue) -1.

3. We compute $A\mathbf{v} = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -3\mathbf{v}$,

we see $\mathbf{v}$ is an eigenvector of A corresponding to (the eigenvalue) -3.

4. We compute $A\mathbf{v} = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 3\mathbf{v}$,

so $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 3.

5. We compute $A\mathbf{v} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 3\mathbf{v}$,

so $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 3.

6. We compute $A\mathbf{v} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{v}$,

so $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 0.

9. As in Example 4.2, we show null$(A - I) \neq \mathbf{0}$ then compute null$(A - I)$ to find $\mathbf{x}$.

Since $A\mathbf{x} = \mathbf{x}$ implies $(A - I)\mathbf{x} = \mathbf{0}$, we have:

$$A - I = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix}$$

Since the columns of $A - I$ are clearly linearly dependent (because $\mathbf{a}_2 = -4\mathbf{a}_1$), the Fundamental Theorem of Invertible Matrices implies that null$(A - I) \neq \mathbf{0}$. That is $A\mathbf{x} = \mathbf{x}$ has a nontrivial solution, so 1 is an eigenvalue of A.

Since $A\mathbf{x} = \mathbf{x}$ implies $(A - I)\mathbf{x} = \mathbf{0}$, we now compute null$(A - I)$.

$$[A - I \mid \mathbf{0}] = \left[\begin{array}{cc|c} -1 & 4 & 0 \\ -1 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1, then $x_1 = 4x_2$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} 4x_2 \\ x_2 \end{bmatrix}$. That is nonzero multiples of $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Q: What does this tell us about null$(A - I)$? What about E_1?

A: The above shows null$(A - I) = \text{span}\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = E_1$, the *eigenspace* of 1.

10. As in Example 4.2, we show null$(A - 4I) \neq \mathbf{0}$ then compute null$(A - 4I)$ to find $\mathbf{x}$.

Since $A\mathbf{x} = 4\mathbf{x}$ implies $(A - 4I)\mathbf{x} = \mathbf{0}$, we have:

$$A - 4I = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Since the columns of $A - 4I$ are clearly linearly dependent (because $\mathbf{a}_2 = -\mathbf{a}_1$), the Fundamental Theorem of Invertible Matrices implies that null$(A - 4I) \neq \mathbf{0}$. That is $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution, so 4 is an eigenvalue of A.

Since $A\mathbf{x} = 4\mathbf{x}$ implies $(A - 4I)\mathbf{x} = \mathbf{0}$, we now compute null$(A - 4I)$.

$$[A - 4I \mid \mathbf{0}] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 4, then $x_2 = x_1$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$. That is nonzero multiples of $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Q: What does this tell us about null$(A - 4I)$? What about E_4?

A: The above shows null$(A - 4I) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = E_4$, the *eigenspace* of 4.

11. As in Example 4.2, we show null$(A+I) \neq \mathbf{0}$ then compute null$(A+I)$ to find $\mathbf{x}$.

Since $A\mathbf{x} = -1\mathbf{x}$ implies $(A+I)\mathbf{x} = \mathbf{0}$, we have:

$$A+I = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

Since the columns of $A+I$ are clearly linearly dependent (because $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$), the Fundamental Theorem of Invertible Matrices implies that null$(A+I) \neq \mathbf{0}$. That is $A\mathbf{x} = -1\mathbf{x}$ has a nontrivial solution, so -1 is an eigenvalue of A.

Since $A\mathbf{x} = -1\mathbf{x}$ implies $(A+I)\mathbf{x} = \mathbf{0}$, we now compute null$(A+I)$.

$$[A+I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue -1, then $x_1 = x_2 = -x_3$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix}$, nonzero multiples of $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Q: What does this tell us about null$(A+I)$? What about E_{-1}?

A: The above shows null$(A+I) = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right) = E_{-1}$, the *eigenspace* of -1.

12. As in Example 4.2, we show null$(A-2I) \neq \mathbf{0}$ then compute null$(A-2I)$ to find $\mathbf{x}$.

Since $A\mathbf{x} = 2\mathbf{x}$ implies $(A-2I)\mathbf{x} = \mathbf{0}$, we have:

$$A-2I = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

Since the columns of $A-2I$ are clearly linearly dependent (because $\mathbf{a}_3 = -\mathbf{a}_2$), the Fundamental Theorem of Invertible Matrices implies that null$(A-2I) \neq \mathbf{0}$. That is $A\mathbf{x} = 2\mathbf{x}$ has a nontrivial solution, so 2 is an eigenvalue of A.

Since $A\mathbf{x} = 2\mathbf{x}$ implies $(A-2I)\mathbf{x} = \mathbf{0}$, we now compute null$(A-2I)$.

$$[A-2I \mid \mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & 2 & -2 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 2, then $x_1 = 0$, $x_3 = x_2$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$, nonzero multiples of $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

15. From the remarks prior to Example 4.4, we have the following key insight:
$\mathbf{x}$ is an eigenvector of A if and only if A transforms $\mathbf{x}$ to a parallel vector.
Why? Because then $A\mathbf{x}$ and $\mathbf{x}$ are multiples of each other. That is, $A\mathbf{x} = \lambda\mathbf{x}$.
Recall that $E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda\} \cup \{\text{the zero vector, } \mathbf{0}\}$.
We have to add the zero vector because eigenvectors are nonzero by definition.

Since $A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, A is the matrix of projection P onto the x-axis.

Consider vectors $\mathbf{v}$ parallel to the x-axis, parallel to the y-axis, and not parallel to either axis.

x-axis: If $\mathbf{v}$ is parallel to the x-axis, P transforms $\mathbf{v}$ to itself. That is, $P(\mathbf{v}) = \mathbf{v}$.
So, all nonzero vectors parallel to the x-axis are eigenvectors of A corresponding to 1.

y-axis: If $\mathbf{v}$ is parallel to the y-axis, P transforms $\mathbf{v}$ to $\mathbf{0}$. That is, $P(\mathbf{v}) = \mathbf{0}$.
So, all nonzero vectors parallel to the y-axis are eigenvectors of A corresponding to 0.

neither: If $\mathbf{v}$ is not parallel to either axis, P transforms $\mathbf{v}$ to a nonparallel vector.
So, all nonzero vectors not parallel to either axis are not eigenvectors of A.

So $E_1 = \text{span}(x\text{-axis}) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $E_0 = \text{span}(y\text{-axis}) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Q: Given that the x-axis is a line, how might we generalize this result?
A: Hint: Consider vectors parallel, perpendicular and neither to the given line.

16. From the remarks prior to Example 4.4, we have the following key insight:
$\mathbf{x}$ is an eigenvector of A if and only if A transforms $\mathbf{x}$ to a parallel vector.
Why? Because then $A\mathbf{x}$ and $\mathbf{x}$ are multiples of each other. That is, $A\mathbf{x} = \lambda\mathbf{x}$.
Recall that $E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda\} \cup \{\text{the zero vector, } \mathbf{0}\}$.
We have to add the zero vector because eigenvectors are nonzero by definition.

From Example 3.59 in Section 3.6, we have: $P_\ell(\mathbf{x}) = \frac{1}{d_1^2+d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix}$.

Since $A\mathbf{x} = \frac{1}{25}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 16x + 12y \\ 12x + 9y \end{bmatrix} = \frac{4}{25}x\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \frac{3}{25}y\begin{bmatrix} 4 \\ 3 \end{bmatrix}$,

A is the matrix of projection P_ℓ onto line ℓ with direction vector $\mathbf{d} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Consider vectors $\mathbf{v}$ parallel to ℓ, perpendicular to ℓ, and neither to direction vector $\mathbf{d}$.

parallel: If $\mathbf{v}$ is parallel to ℓ, P_ℓ transforms $\mathbf{v}$ to itself. That is, $P_\ell(\mathbf{v}) = \mathbf{v}$.
So, all nonzero vectors parallel to ℓ are eigenvectors of A corresponding to 1.

perpendicular: If $\mathbf{v}$ is perpendicular to ℓ, P_ℓ transforms $\mathbf{v}$ to $\mathbf{0}$. That is, $P_\ell(\mathbf{v}) = \mathbf{0}$.
So, all nonzero vectors perpendicular to ℓ are eigenvectors of A corresponding to 0.

neither: If $\mathbf{v}$ is neither parallel nor perpendicular, P_ℓ transforms $\mathbf{v}$ to a nonparallel vector.
So, all vectors not parallel or perpendicular to ℓ are not eigenvectors of A.

$E_1 = \text{span}(\text{parallel to } \mathbf{d}) = \text{span}\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)$

$E_0 = \text{span}(\text{perpendicular to } \mathbf{d}) = \text{span}\left(\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right)$.

17. From the remarks prior to Example 4.4, we have the following key insight:
x is an eigenvector of A if and only if A transforms **x** to a parallel vector.
Why? Because then $A\mathbf{x}$ and $\mathbf{x}$ are multiples of each other. That is, $A\mathbf{x} = \lambda\mathbf{x}$.
Recall that $E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda\} \cup \{\text{the zero vector, } \mathbf{0}\}$.
We have to add the zero vector because eigenvectors are nonzero by definition.

Since $A\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$, A is the matrix of stretching S.

Consider vectors **v** parallel to the x-axis, parallel to the y-axis, and not parallel to either axis.

x-axis: If **v** is parallel to the x-axis, S transforms **v** to twice itself. That is, $S(\mathbf{v}) = 2\mathbf{v}$.
So, all nonzero vectors parallel to the x-axis are eigenvectors of A corresponding to 2.

y-axis: If **v** is parallel to the y-axis, S transforms **v** to thrice itself. That is, $S(\mathbf{v}) = 3\mathbf{v}$.
So, all nonzero vectors parallel to the y-axis are eigenvectors of A corresponding to 3.

neither: If **v** is not parallel to either axis, S transforms **v** to a nonparallel vector.
So, all vectors not parallel to either axis are not eigenvectors of A.

So $E_2 = \text{span}(x\text{-axis}) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $E_3 = \text{span}(y\text{-axis}) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Q: Following this exact same process, how might we generalize this result?

A: If $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, then $E_a = \text{span}(x\text{-axis})$ and $E_d = \text{span}(y\text{-axis})$.

18. From the remarks prior to Example 4.4, we have the following key insight:
x is an eigenvector of A if and only if A transforms **x** to a parallel vector.
Why? Because then $A\mathbf{x}$ and $\mathbf{x}$ are multiples of each other. That is, $A\mathbf{x} = \lambda\mathbf{x}$.
Recall that $E_\lambda = \text{null}(A - \lambda I) = \{\text{eigenvectors of } \lambda\} \cup \{\text{the zero vector, } \mathbf{0}\}$.
We have to add the zero vector because eigenvectors are nonzero by definition.

From Example 3.58 in Section 3.6, we have: $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$A\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$, so A is the matrix of rotation R_{90°.

Consider the zero vector **0** and all nonzero vectors **v**.

$\mathbf{v} = \mathbf{0}$: Since a rotation leaves the zero vector fixed, $R_{90^\circ}(\mathbf{0}) = \mathbf{0}$.
However, the zero vector is not an eigenvector of A corresponding to 0.
Why not? Because the zero vector is *zero* and eigenvectors must be nonzero by definition.

$\mathbf{v} \neq \mathbf{0}$: A rotation transforms any nonzero vector to a nonparallel vector.
So, all nonzero vectors are not eigenvectors of A when A is the matrix of any rotation.

Q: Is it still true that $E_0 = \text{span}(\mathbf{0}) = \mathbf{0}$?

A: Yes, because $E_0 = \{\text{eigenvectors of } \lambda\} \cup \{\text{the zero vector, } \mathbf{0}\} = \{\mathbf{0}\}$.

21. From the remarks prior to Example 4.4, we have the following key insight:
x is an eigenvector of A if and only if A transforms **x** to a parallel vector.
So, lines that do *not* bend at the unit circle represent eigenvectors.
The extension beyond the circle tells us if the vector has been stretched.

Since the lines do not bend on the line ℓ $y = -x$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

So, we consider vectors **v** parallel to **d**, perpendicular to **d**, and neither.

parallel: On the line $y = x$, the lines do not bend and extend precisely 2 units beyond it. So:
If **v** is parallel to the **d**, S transforms **v** to thrice itself. That is, $S(\mathbf{v}) = 2\mathbf{v}$.
So, all nonzero vectors parallel to **d** are eigenvectors of A corresponding to 2.

perp: On the line $y = -x$, the lines extend precisely 0 units beyond the unit circle. So:
If **v** is perpendicular to the **d**, S transforms **v** to **0**. That is, $S(\mathbf{v}) = 0\mathbf{v}$.
So, all nonzero vectors perpendicular to **d** are eigenvectors of A corresponding to 0.

neither: Off the lines $y = x$ and $y = -x$, the lines *do* bend at the unit circle. So:
If **v** is not parallel or perpendicular to **d**, S transforms **v** to a nonparallel vector.
So, all vectors not parallel or perpendicular to **d** are not eigenvectors of A.

So $E_2 = \text{span}(\mathbf{d}) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $E_0 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

22. From the remarks prior to Example 4.4, we have the following key insight:
x is an eigenvector of A if and only if A transforms **x** to a parallel vector.
So, lines that do *not* bend at the unit circle represent eigenvectors.
Here, however, all lines bend at the unit circle, so we conclude there are no eigenvectors.

Q: What types of transformations have no eigenvectors?
A: Rotations. See Exercise 18. Is the graph in Exercise 22 suggestive of a rotation?

35. (a) To find the eigenvalues of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we solve $\det(A - \lambda I) = 0 \Leftrightarrow$

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - \operatorname{tr}(A)\lambda + \det A = 0.$$

(b) Using the quadratic formula, the solutions to the equation in part (a) are

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{a+d \pm \sqrt{a^2+d^2+2ad-4ad+4bc}}{2}$$
$$= \frac{1}{2}\left(a+d \pm \sqrt{(a-d)^2+4bc}\right).$$

(c) Let $\lambda_1 = \frac{1}{2}(a+d) + \sqrt{(a-d)^2+4bc}))$ and $\lambda_2 = \frac{1}{2}(a+d) - \sqrt{(a-d)^2+4bc}))$.
So, $\lambda_1 + \lambda_2 = \frac{1}{2}(a+d) + \frac{1}{2}(a+d) = a+d = \operatorname{tr}(A)$.
Also, $\lambda_1\lambda_2 = \frac{1}{4}[(a+d)^2 - ((a-d)^2+4bc)] = \frac{1}{4}[4ad-4bc] = ad-bc = \det A$.

36. (a) If A is to have two distinct real eigenvalues, the discriminant of the equation in Exercise 35(b) must be positive. That is, $(a-d)^2 + 4bc > 0 \Leftrightarrow (a-d)^2 > -4bc$.
If $a = d$, then neither b nor c can equal zero if this inequality is to hold.
If $a \neq d$, whenever b and c have the same sign this inequality holds.

(b) If A is to have one real eigenvalue, the discriminant must be zero.
That is, $(a-d)^2 + 4bc = 0 \Leftrightarrow (a-d)^2 = -4bc$.
If $a = d$, then neither b or c must be zero if this inequality is to hold.

(c) If A is to have no real eigenvalue, the discriminant must be negative.
That is, $(a-d)^2 + 4bc < 0 \Leftrightarrow (a-d)^2 < -4bc$.
If $a = d$, then neither b nor c can equal zero if this inequality is to hold.
If $a \neq d$, then b and c must have opposite signs if this inequality is to hold.

37. As in Example 4.5, we find all solutions λ of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ 0 & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + ad$$

Since $\lambda^2 - (a+d)\lambda + ad = (\lambda - a)(\lambda - d) = 0$, the solutions are $\lambda = a, d$.

$\lambda = a$: $A - aI = \begin{bmatrix} a-a & b \\ 0 & a-d \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & a-d \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to a, $x_1 = t$, $x_2 = 0$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} t \\ 0 \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So, $E_a = \text{null}(A - aI) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

$\lambda = d$: $A - dI = \begin{bmatrix} a-d & b \\ 0 & d-d \end{bmatrix} = \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to d, $(d-a)x_1 = bx_2 = (d-a)bt$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} bt \\ (d-a)t \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} b \\ d-a \end{bmatrix}$.

So, $E_b = \text{null}(A - bI) = \text{span}\left(\begin{bmatrix} b \\ d-a \end{bmatrix}\right)$.

38. As in Example 4.5, we find all solutions λ of the equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ -b & a-\lambda \end{bmatrix} = \lambda^2 - 2a\lambda + a^2 + b^2$$

Since $\lambda^2 - 2a\lambda + a^2 + b^2 = 0$ implies $\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$, we have $\lambda = a + bi, a - bi$.

$a + bi$: $A - (a+bi)I = \begin{bmatrix} a-(a+bi) & b \\ -b & a-(a+bi) \end{bmatrix} = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \longrightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to a, $x_2 = ix_1$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} t \\ it \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

So, $E_{a+bi} = \text{null}(A - (a+bi)I) = \text{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$.

$a - bi$: $A - (a-bi)I = \begin{bmatrix} a-(a-bi) & b \\ -b & a-(a-bi) \end{bmatrix} = \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} \longrightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to a, $x_2 = -ix_1$.

These eigenvectors are of the form $\mathbf{x} = \begin{bmatrix} t \\ -it \end{bmatrix}$, nonzero multiples of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

So, $E_{a-bi} = \text{null}(A - (a-bi)I) = \text{span}\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$.