Chapter 3 Review

1. We will explain and give counter examples to justify our answers below.

   (a) ***True.*** See Exercise 28 in Section 3.2.

   Let A be an $m \times n$ matrix, then A^T is an $n \times m$ matrix.
   Since AA^T is $[m \times n][n \times m]$, AA^T is an $m \times m$ matrix.
   Since A^TA is $[n \times m][m \times n]$, A^TA is an $n \times n$ matrix.
   So not only are AA^T and A^TA defined, they are square matrices.

   (b) ***False.*** See Theorem 3.27 of Section 3.5.

   Theorem 3.27 of Section 3.5 implies $AB = O \Rightarrow B = O$ if and only if A is invertible.
   Since vectors are matrices, when $AB = O \Rightarrow B = O$, what do we know about null(A)?
   Since $B = O$ means null(A) $= \mathbf{0}$, Theorem 3.27 of Section 3.5 implies A is invertible.
   When A^{-1} exists, we have: $AB = O \Rightarrow (A^{-1}A)B = A^{-1}O = O \Rightarrow B = O$.
   As a counterexample, consider: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   (c) ***False.*** See Exercises 20 through 23 in Section 3.3.

   What is the source of the problem? Matrix multiplication does *not* commute.
   Instead, we should solve for X using A^{-1}.
   If $XA = B$ then $X(AA^{-1}) = BA^{-1}$, so $X = BA^{-1}$ (not always equal to $A^{-1}B$).

   When $X = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$,

   then $A^{-1}B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = BA^{-1}$.

   So, $XA = B \Rightarrow X = A^{-1}B$ if and only if $A^{-1}B = BA^{-1}$.
   Does $XA = AX$ if and only if $A^{-1}B = BA^{-1}$? Why or why not?

   (d) ***True.*** See Theorem 3.11 in Section 3.3.

   What is the idea? E performs exactly 1 operation and E^{-1} undoes it.
   For example, if E performs kR_i, then E^{-1} performs $\frac{1}{k}R_i$.
   If E performs $R_i + R_j$, then E^{-1} performs $R_i - R_j$.
   Let $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $E^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Confirm this by showing $EE^{-1} = I$.

   (e) ***True.*** See Section 3.3. Prove this by considering each type of E separately.

   Since E is obtained from I, we need only consider the entries that differ from I.

   If E performs kR_i, then $[E]_{ii} = k = [E^T]_{ii}$, so $E^T = E$ performs kR_i.
   If E performs $R_i + kR_j$, then $[E]_{ij} = k = [E^T]_{ji}$.
   So, E^T is an elementary matrix that performs $R_j + kR_i$.
   If E performs $R_i \leftrightarrow R_j$, then $[E]_{ii} = [E]_{jj} = 0$ and $[E]_{ij} = [E]_{ji} = 1$.
   So E^T performs $R_i \leftrightarrow R_j$, too since $[E^T]_{ii} = [E]_{ii}$ and $[E^T]_{ji} = [E]_{ij}$.

   So, when E performs kR_i or $R_i \leftrightarrow R_j$, then $E^T = E$.
   Furthermore, when E performs $R_i \leftrightarrow R_j$, $E^T = E^{-1} = E$.

1. We explain and give counter examples to justify our answers below (continued).

(f) ***False***. See Section 3.3.

An elementary matrix can only perform *one* elementary row operation on I.
But E_2E_1 performs *two* elementary row operations on I.
When $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, then $E_2E_1 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$.

(g) ***True***. See Theorem 3.21 in Section 3.5.

We should be able to easily recreate the proof in the text.
Is the following enough? If $\mathbf{u}$, $\mathbf{v}$ are in null(A), then
$A(c\mathbf{u} + d\mathbf{v}) = c(A\mathbf{u}) + d(A\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ is in null(A).

(h) ***False***. See the discussions of Exercises 1 through 6 in Section 3.5.

What is the problem? Every subspace must contain the zero vector.
What condition do we need to add make the statement true?
Every plane *that passes through the origin* is a subspace.
Is the dimension of the plane actually two?
Yes. A basis for the subspace is the two direction vectors in the parametric form.
For that form, $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, and further discussion see Section 1.3.

(i) ***True***. See the definition of a linear transformation in Section 3.6.

We verify that $T(\mathbf{x}) = -\mathbf{x}$ satisfies the two necessary conditions.

$T(\mathbf{u}+\mathbf{v}) = -(\mathbf{u}+\mathbf{v}) = (-\mathbf{u}) + (-\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

$T(c\mathbf{u}) = -(c\mathbf{u}) = c(-\mathbf{u}) = cT(\mathbf{u})$

Theorems 3.30 and 3.31 of Section 3.6 imply an A with $A\mathbf{x} = -\mathbf{x}$ is also enough.
Note that $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ does just that.
Finally, note that $A = -I$. Why does that make sense?

(j) ***False***. See Theorems 3.30 and 3.31 of Section 3.6.

The matrix A must be 5×4 not 4×5.

13. Find bases for row(A), col(A), and null(A) by Examples 3.45, 3.47, and 3.48 in Section 3.5.

row(A): A basis for row(A) must span the rows of A and be linearly independent.
The linearly independent rows (which are simply the nonzero rows) of U do just that.

$$\text{Since } A = \begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 1 & -2 & 2 & 3 & 1 \\ 4 & -8 & 3 & 2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = U$$

we conclude $\{[\,2\ \ -4\ \ 5\ \ 8\ \ 5\,], [\,0\ \ 0\ \ 1\ \ 2\ \ 3\,], [\,0\ \ 0\ \ 0\ \ 0\ \ 1\,]\}$ is a basis for row(A).

Q: In $A \longrightarrow U$, why is sufficient to reduce A only to row echelon form U?

A: As the remark after Example 3.46 in Section 3.5 explains and demonstrates, the nonzero rows of U are linearly independent. That is all that is required. Why?

We should also note that provided $A \longrightarrow U$ uses no row interchanges, the corresponding rows in A are also linearly independent.
Whence, it is obvious that those rows form a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A \longrightarrow U$, the columns with leading entries in U are linearly independent.
As in Example 3.47, the corresponding columns in A are also linearly independent.
Whence, it is obvious that those columns form a basis for col(A).

$$\text{Since } A = \begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 1 & -2 & 2 & 3 & 1 \\ 4 & -8 & 3 & 2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$\text{we conclude that } \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\} \text{ is a basis for col}(A).$$

null(A): Since $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v}$ is in null(A), we solve $[\,A\,|\,\mathbf{0}\,] \longrightarrow [\,R\,|\,\mathbf{0}\,]$ to find the conditions.

$$\text{Note } A = \begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 1 & -2 & 2 & 3 & 1 \\ 4 & -8 & 3 & 2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R$$

$$[\,R\,|\,\mathbf{0}\,] = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= 2s + 1t \\ x_2 &= s \\ x_3 &= -2t \\ x_4 &= t \\ x_5 &= 0 \end{aligned}$$

$$\text{Therefore, } \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for null}(A).$$

14. Given $A \longrightarrow B$, we compare row(A) to row(B) and col(A) to col(B) separately below.

row: By Theorem 3.20 in Section 3.5, if $A \longrightarrow B$ then row(A) = row(B).
Observe that the rows in A are linearly combinations the rows in B and vice versa.
See proof of Theorem 3.20 in Section 3.5 for details.

col: Since $A \longrightarrow B$ does *not* use linear combinations of columns, we suspect col(A) $\neq$ col(B).
Furthermore, recall a basis for col(A) is taken directly from A (see Exercise 13 above).
We prove $A \longrightarrow B$ does *not* imply col(A) = col(B) with the following counterexample.

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $A \overset{R_3 - R_1}{\longrightarrow} B$.

Furthermore, $\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ and $\text{col}(B) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

Then $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in col(A), but $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is *not* in col(B), so col(A) $\neq$ col(B).

Q: If $A \longrightarrow B$ and $\mathbf{A}_i \neq \mathbf{0} \longrightarrow \mathbf{B}_i = \mathbf{0}$, does col($A$) = col($B$)?
A: No, this is the generalization of the situation in the example above.
What is the problem? The ith component of the basis vectors for col(B) are all zero, but ith component of the basis vectors for col(A) are not all zero.

Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Does col($A$) = col($B$)?

15. We consider when A *is* invertible and when A is *not* invertible separately below.

A^{-1}: By Theorem 3.9 in Section 3.3, if A is invertible then so is A^T.
Therefore, by Theorem 3.27 of Section 3.5, null(A) = null(A^T) = $\mathbf{0}$.
This conclusion is not stated directly but following immediately from c. or g. Why?
From c.: If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $\mathbf{x} = \mathbf{0}$ = null(A).
From g.: If nullity(A) = 0 then $\mathbf{0}$ is a basis for null(A) which implies null(A) = $\mathbf{0}$.

no A^{-1}: Since the column space is affected by row reduction, we suspect null(A) $\neq$ null(A^T).
We prove null(A) $\neq$ null(A^T) with the following counterexample.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Note $A^T \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then, $\text{null}(A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$ and $\text{null}(A^T) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

Then $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is in null(A), but $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is *not* in null(A^T), so null(A) $\neq$ null(A^T).

16. If $\sum \mathbf{A}_i = \mathbf{0}$, then the rows of A are linearly dependent.
Therefore, by Theorem 3.27 in Section 3.5, A is not invertible.

Note this is the contrapositive of a. implies g. which states:
If A is invertible, then the rows of A are linearly independent.

Q: This also gives us a nontrivial solution to $A^T\mathbf{x} = \mathbf{0}$. Which one?
A: Let $\mathbf{x}$ be the vector each of whose components is equal to 1. For example:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Q: Does $A^T\mathbf{x} = \mathbf{0}$ having a nontrivial solution imply A is not invertible?
A: Yes. If $A^T\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A^T is not invertible.
That implies A is not invertible.

17. We consider A^TA and AA^T separately below.

A^TA: Since A has n linearly independent columns, $\text{rank}(A) = n$.
By Theorem 3.28 of Section 3.5, $\text{rank}(A^TA) = \text{rank}(A) = n$, so A^TA is invertible.

AA^T: Since m may be greater than n, we should suspect that AA^T is not necessarily invertible.
We prove AA^T is not necessarily invertible with the following counterexample.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is obviously invertible.

On the other hand, $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Since $\text{rank}(AA^T) = 2 < 3$, AA^T is *not* invertible.
Q: In this case, we have $\text{rank}(AA^T) \le \text{rank}(A)$. Is that always true?
A: Yes, since we saw in Exercise 57 of Section 3.5 that $\text{rank}(AB) \le \text{rank}(A)$.

Q: In this case, we saw $\text{rank}(AA^T) = \text{rank}(A^TA)$. Is that always true?
A: Hint: Consider the proof of Theorem 3.28 in Section 3.5.

18. We find the matrix $[T]$ and the formula for the linear transformation T separately below.

matrix: By Theorem 3.31 of Section 3.6, $[T] = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}$. Therefore, we have:

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\left(\tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\left(\tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \tfrac{1}{2}\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So, the matrix of the linear transformation is $[T] = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$.

formula: Since $T(\mathbf{v}) = [T]\mathbf{v}$, the formula for T is $\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 4x - y \end{bmatrix}$.
We could have found the formula directly by using $\mathbf{v}$ instead of $\mathbf{e}_1$ and $\mathbf{e}_2$.

19. We find rotation $[R]$, projection $[P]$. Then by Theorem 3.32 of Section 3.6, $[P \circ R] = [P][R]$.

$[R]$: From Example 3.58 in Section 3.6, the rotation $[R]$ has matrix: $[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

So, a rotation of 45° is given by $[R] = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

$[P]$: From Example 3.59 in Section 3.6, $[P]$ through ℓ with $\mathbf{d}$ is: $[P] = \frac{1}{d_1^2+d_1^2}\begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$.

So, for $y = -2x$ with $\mathbf{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ we have: $[P] = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$.

So $[P][R] = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}-2\sqrt{2}}{10} & \frac{-\sqrt{2}-2\sqrt{2}}{10} \\ \frac{-2\sqrt{2}+4\sqrt{2}}{10} & \frac{2\sqrt{2}+4\sqrt{2}}{10} \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{2}}{10} & \frac{-3\sqrt{2}}{10} \\ \frac{\sqrt{2}}{5} & \frac{3\sqrt{2}}{5} \end{bmatrix}$.

20. To prove $\mathbf{v}$ and $T(\mathbf{v})$ are linearly independent, we need to show the following:
If $c\mathbf{v} + dT(\mathbf{v}) = \mathbf{0}$, then $c = d = 0$.

$$c\mathbf{v} + dT(\mathbf{v}) = \mathbf{0} \overset{\substack{T \text{ is}\\ \text{linear}}}{\Rightarrow} T(c\mathbf{v} + dT(\mathbf{v})) = T(\mathbf{0}) = \mathbf{0} \overset{\substack{T \text{ is}\\ \text{linear}}}{\Rightarrow}$$

$$cT(\mathbf{v}) + dT^2(\mathbf{v}) = \mathbf{0} \overset{\substack{T^2(\mathbf{v})=\mathbf{0}\\ \text{given}}}{\Rightarrow} cT(\mathbf{v}) = \mathbf{0} \overset{\substack{T(\mathbf{v})\neq\mathbf{0}\\ \text{given}}}{\Rightarrow} c = 0$$

So: $c\mathbf{v} + dT(\mathbf{v}) = \mathbf{0} \overset{\substack{c=0\\ \text{above}}}{\Rightarrow} dT(\mathbf{v}) = \mathbf{0} \overset{\substack{T(\mathbf{v})\neq\mathbf{0}\\ \text{given}}}{\Rightarrow} d = 0$, as we were to show.

Note: The key to this proof was using the givens: $T(\mathbf{v}) \neq \mathbf{0}$ and $T^2(\mathbf{v}) = \mathbf{0}$.