17. We find bases for row(A), col(A), and null(A) as in Examples 3.45, 3.47, and 3.48 respectively.

row(A): A basis for row(A) must span the rows of A and be linearly independent.
Given $A \longrightarrow R$, Theorem 3.20 asserts that the rows of R span the rows of A. Why?
Because the rows of A are linear combinations of the rows of R (and vice-versa).
Finally, we simply observe that the nonzero rows of R are linearly independent.

$$\text{Since } A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R,$$

we conclude that $\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}\}$ is a basis for row(A).
We should also note that provided $A \longrightarrow R$ uses no row interchanges, the corresponding rows in A are also linearly independent.
Whence, it is obvious that those rows form a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As shown in Example 3.47, the corresponding columns in A are also linearly independent.
Whence, it is obvious that those columns form a basis for col(A).

$$\text{Since } A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R,$$

we conclude that $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A).

null(A): Since $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v}$ is in null(A), we solve $[\, A \,|\, \mathbf{0} \,] \longrightarrow [\, R \,|\, \mathbf{0} \,]$ to find the conditions:

$$[\, R \,|\, \mathbf{0} \,] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 \; free \end{array} \Rightarrow \begin{array}{rcr} x_1 & = & 1s \\ x_2 & = & -2s \\ x_3 & = & 1s \end{array}$$

Since t is arbitrary, null(A) = span $\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$. So, $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for null(A).

18. We find bases for row(A), col(A), and null(A) as in Examples 3.45, 3.47, and 3.48 respectively.

row(A): A basis for row(A) must span the rows of A and be linearly independent.
Given $A \longrightarrow U$, Theorem 3.20 asserts that the rows of U span the rows of A. Why?
Because the rows of A are linear combinations of the rows of U (and vice-versa).
Finally, we simply observe that the nonzero rows of U are linearly independent.

$$\text{Since } A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \xrightarrow{R_3 - R_1 + R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U,$$

we conclude that $\{[\,1 \;\; 1 \;\; -3\,], [\,0 \;\; 2 \;\; 1\,]\}$ is a basis for row(A).

Q: In $A \longrightarrow U$, why is it sufficient to reduce A only to row echelon form U?

A: As the remark following Example 3.46 explains and then demonstrates by example, the nonzero rows of U are linearly independent. That is all that is required. Why?

We should also note that provided $A \longrightarrow U$ uses no row interchanges,
the corresponding rows in A are also linearly independent.
Whence, it is obvious that those rows form a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A \longrightarrow U$, the columns with leading entries in U are linearly independent.
As in Example 3.47, the corresponding columns in A are also linearly independent.
Whence, it is obvious that those columns form a basis for col(A).

$$\text{Since } A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U,$$

$$\text{we conclude that } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ is a basis for col}(A).$$

null(A): Since $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v}$ is in null(A), we solve $[\,A \,|\, \mathbf{0}\,] \longrightarrow [\,U' \,|\, \mathbf{0}\,]$ to find the conditions.
We row reduce U one more step to U' make it easier to find the conditions:

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \xrightarrow{R_3 - R_1 + R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 3R_2} \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U'$$

$$[\,U' \,|\, \mathbf{0}\,] = \left[\begin{array}{ccc|c} 1 & 7 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 7x_2 = 0 \\ 2x_2 + x_3 = 0 \\ x_2 \textit{ free} \end{array} \Rightarrow \begin{array}{rcl} x_1 & = & -7s \\ x_2 & = & s \\ x_3 & = & -2s \end{array}$$

$$\text{Since } s \text{ is arbitrary, null}(A) = \text{span}\left(\begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix} \right). \text{ So, } \left\{ \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix} \right\} \text{ is a basis for null}(A).$$

21. We find bases for row(A) and col(A) following Examples 3.45 and 3.47 respectively.

row(A): A basis for col(A) must span the columns of A and be linearly independent.
Clearly, the linearly independent *columns* of A^T do just that.
When $A^T \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As in Example 3.47, the corresponding columns in A^T are also linearly independent.
Whence, it is obvious that the *transposes* of those columns form a basis for row(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

we conclude that $\left\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right\}$ is a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A^T \longrightarrow R$, the linearly independent *rows* (the nonzero rows) of R do just that.
Whence, it is obvious that the *transposes* of those rows form a basis for col(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

we conclude that $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A).

We should also note that provided $A^T \longrightarrow R$ uses no row interchanges,
the corresponding rows in A^T are also linearly independent.
Whence, it is obvious that the *transposes* of those rows form a basis for col(A).

22. We find bases for row(A) and col(A) following Examples 3.45 and 3.47 respectively.

row(A): A basis for col(A) must span the columns of A and be linearly independent.
Clearly, the linearly independent *columns* of A^T do just that.
When $A^T \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As in Example 3.47, the corresponding columns in A^T are also linearly independent.
Whence, it is obvious that the *transposes* of those columns form a basis for row(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R,$$

we conclude that $\left\{\begin{bmatrix} 1 & 1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}\right\}$ is a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A^T \longrightarrow R$, the linearly independent *rows* (the nonzero rows) of R do just that.
Whence, it is obvious that the *transposes* of those rows form a basis for col(A).

$$\text{Since } A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ -3 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R,$$

we conclude that $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a basis for col(A).

24. We find bases for row(A) and col(A) following Examples 3.45 and 3.47 respectively.

row(A): A basis for col(A) must span the columns of A and be linearly independent.
Clearly, the linearly independent *columns* of A^T do just that.
When $A^T \longrightarrow R$, the columns with leading 1s in R are linearly independent.
As in Example 3.47, the corresponding columns in A^T are also linearly independent.
Whence, it is obvious that the *transposes* of those columns form a basis for row(A).

$$\text{Since } A^T = \begin{bmatrix} 2 & -1 & 1 \\ -4 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R,$$

we conclude that $\left\{ \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 1 & 2 & 3 \end{bmatrix} \right\}$ is a basis for row(A).

col(A): A basis for col(A) must span the columns of A and be linearly independent.
When $A^T \longrightarrow R$, the linearly independent *rows* (the nonzero rows) of R do just that.
Whence, it is obvious that the *transposes* of those rows form a basis for col(A).

$$\text{Since } A^T = \begin{bmatrix} 2 & -1 & 1 \\ -4 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R,$$

we conclude that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for col(A).

39. If nullity$(A) > 0$, then the columns of A are linearly dependent.
Though we could prove this using theorems, it is instructive to prove it directly.

If nullity$(A) > 0$, then there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$.

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{x}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$.

Since $\mathbf{x} \neq \mathbf{0}$ at least one $c_i \neq 0$. Then $A\mathbf{x} = \sum c_i \mathbf{a}_i = \mathbf{0}$ where at least one $c_i \neq 0$.
Therefore, the columns of A are linearly dependent.

So all we have to show is: If A is a 3×5 matrix, then nullity$(A) > 0$.

$$\text{nullity}(A) \overset{\substack{\text{Rank}\\ \text{Thm}}}{=} n - \text{rank}(A) \overset{\substack{A\text{ is}\\ 3\times 5}}{=} 5 - \text{rank}(A) \overset{\text{rank}(A)\leq 3}{\geq} 5 - 3 = 2 > 0$$

Q: If A is 3×5, why is it obvious that rank$(A) \leq 3$?
A: Recall, that the number of vectors in a basis for row(A) = dim(row(A)).
Now note the rows contain a basis for row(A), so dim(row(A)) $\leq$ number of rows.
So, rank(A) = dim(row(A)) $\leq$ the number of rows = 3.

40. If the number of rows > the number of columns, then the rows are linearly dependent.
Why? Rank must satisfy the following two conditions simultaneously:
1) rank(A) = dim(row(A)) $\leq$ the number of rows
2) rank(A) = dim(col(A)) $\leq$ the number of columns
Therefore, rank must be less than or equal to the smaller of these two numbers.

So, dim(row(A)) = rank(A) $\leq$ the number of columns < the number of rows.
That is, dim(row(A)) = the number of vectors in a basis for row(A) < the number of rows.
Recall, we can find a basis for row(A) from among the rows of A.
Therefore, the fact that dim(row(A)) < the number of rows implies the following:
There exists at least one row that is a linear combination of the remaining rows.
That is, the rows of A are linearly dependent.

If A is a 4×2 matrix, then the rows are linearly dependent.
This is now obvious because the number of rows = $4 > 2$ = the number of columns.

Q: When we compare this result to Theorem 2.8 of Section 2.3, what do we notice?

Q: If the number of columns > the number of rows, are the columns are linearly dependent?

41. Rank must satisfy the following two conditions simultaneously:
1) rank(A) = dim(row(A)) $\leq$ the number of rows
2) rank(A) = dim(col(A)) $\leq$ the number of columns
Therefore, rank must be less than or equal to the smaller of these two numbers.

Since A is 3×5, rank(A) can equal 0, 1, 2, or 3.
Therefore, since $n = 5$ and nullity$(A) = n -$ rank(A), we have:

nullity$(A) = 5 - 3 = 2$, $5 - 2 = 3$, $5 - 1 = 4$, or $5 - 0 = 5$

42. Since A is 4×2, rank(A) can equal 0, 1, or 2.
Therefore, since $n = 2$ and nullity$(A) = n -$ rank(A), we have

nullity$(A) = 2 - 2 = 0$, $2 - 1 = 1$, or $2 - 0 = 2$.

43. $A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \xrightarrow[R_3 - aR_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & a \\ 0 & a+1 & \frac{a+1}{2} \\ 0 & 0 & \frac{(a+1)(a-2)}{2} \end{bmatrix}$.

If $a = -1$, then $A \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 1$.

If $a = 2$, then $A \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$. Otherwise, $\text{rank}(A) = 3$.

44. $A = \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -2 & -1 & a \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 2 & a-2 \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix} \longrightarrow$

$$\begin{bmatrix} 1 & 2 & a-2 \\ 0 & 1 & a - \frac{4}{3} \\ 0 & 2-2a & -(a-1)^2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & a-2 \\ 0 & 1 & a - \frac{4}{3} \\ 0 & 0 & (a-1)(a-\frac{5}{3}) \end{bmatrix} \Rightarrow$$

If $a = 1, \frac{5}{3}$, then $A \longrightarrow \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$. Otherwise, $\text{rank}(A) = 3$.

45. As in Example 3.52, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^3 \Leftrightarrow$ When $A = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$, $\text{rank}(A) = 3$.

$$A = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $\text{rank}(A) = 3$, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^3$.

46. As in Example 3.52, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^3 \Leftrightarrow$ When $A = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$, $\text{rank}(A) = 3$.

$$A = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 5 & -3 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 + R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 4 & -2 \end{bmatrix}$$

Since $\text{rank}(A) < 3$, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ does *not* form a basis for $\mathbb{R}^3$.

47. As in Example 3.52, $\{\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^4 \Leftrightarrow$ When $A = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$, rank$(A) = 4$.

$$A = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3+R_2+R_4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Since rank$(A) = 4$, $\{\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^4$.

48. As in Example 3.52, $\{\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^4 \Leftrightarrow$ When $A = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$, rank$(A) = 4$.

$$A = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_4+R_1+R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Since rank$(A) = 4$, $\{\mathbf{x}, \mathbf{v}, \mathbf{w}\}$ form a basis for $\mathbb{R}^4$.

49. We find the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$ by finding c_1 and c_1 such that $\mathbf{w} = c_1\mathbf{b}_1 + c_1\mathbf{b}_2$.
As in Example 2.18 of Section 2.3, we form the matrix $A = [\, \mathbf{b}_1 \;\; \mathbf{b}_2 \,|\, \mathbf{w} \,]$ and row reduce.

$$A = [\, \mathbf{b}_1 \;\; \mathbf{b}_2 \,|\, \mathbf{w} \,] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Since $\mathbf{w} = 3\mathbf{b}_1 - 2\mathbf{b}_2$, we have the coordinate vector $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

50. We find the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$ by finding c_1 and c_1 such that $\mathbf{w} = c_1\mathbf{b}_1 + c_1\mathbf{b}_2$.
As in Example 2.18 of Section 2.3, we form the matrix $A = [\, \mathbf{b}_1 \;\; \mathbf{b}_2 \,|\, \mathbf{w} \,]$ and row reduce.

$$A = [\, \mathbf{b}_1 \;\; \mathbf{b}_2 \,|\, \mathbf{w} \,] = \left[\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

Since $\mathbf{w} = 7\mathbf{b}_1 - 4\mathbf{b}_2$, we have the coordinate vector $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

51. We row reduce over $\mathbb{Z}_2$ to find rank(A), then nullity$(A) = n -$ rank(A).

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Since rank$(A) = 2$, we have nullity$(A) = 3 - 2 = 1$.