17. We find bases for \( \text{row}(A) \), \( \text{col}(A) \), and \( \text{null}(A) \) as in Examples 3.45, 3.47, and 3.48 respectively.

\( \text{row}(A) \): A basis for \( \text{row}(A) \) must span the rows of \( A \) and be linearly independent.
Given \( A \rightarrow R \), Theorem 3.20 asserts that the rows of \( R \) span the rows of \( A \). Why?
Because the rows of \( A \) are linear combinations of the rows of \( R \) (and vice-versa).
Finally, we simply observe that the nonzero rows of \( R \) are linearly independent.

Since \( A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R \),
we conclude that \( \{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \} \) is a basis for \( \text{row}(A) \).

We should also note that provided \( A \rightarrow R \) uses no row interchanges,
the corresponding rows in \( A \) are also linearly independent.

Whence, it is obvious that those rows form a basis for \( \text{row}(A) \).

\( \text{col}(A) \): A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent.
When \( A \rightarrow R \), the columns with leading 1s in \( R \) are linearly independent.
As shown in Example 3.47, the corresponding columns in \( A \) are also linearly independent.

Whence, it is obvious that those columns form a basis for \( \text{col}(A) \).

Since \( A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R \),
we conclude that \( \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \) is a basis for \( \text{col}(A) \).

\( \text{null}(A) \): Since \( Av = 0 \) implies \( v \) is in \( \text{null}(A) \), we solve \( [A \mid 0] \rightarrow [R \mid 0] \) to find the conditions:

\[
\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{cases} x_1 - x_3 = 0 & x_1 = 1s \\ x_2 + 2x_3 = 0 & x_2 = -2s \\ x_3 \text{ free} & x_3 = 1s \end{cases}
\]

Since \( t \) is arbitrary, \( \text{null}(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \). So, \( \{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \} \) is a basis for \( \text{null}(A) \).
18. We find bases for \( \text{row}(A) \), \( \text{col}(A) \), and \( \text{null}(A) \) as in Examples 3.45, 3.47, and 3.48 respectively.

**row}(A): A basis for \( \text{row}(A) \) must span the rows of \( A \) and be linearly independent.

Given \( A \rightarrow U \), Theorem 3.20 asserts that the rows of \( U \) span the rows of \( A \). Why?

Because the rows of \( A \) are linear combinations of the rows of \( U \) (and vice-versa).

Finally, we simply observe that the nonzero rows of \( U \) are linearly independent.

\[
A = \begin{bmatrix}
1 & 1 & -3 \\
0 & 2 & 1 \\
1 & -1 & -4
\end{bmatrix}
\quad \xrightarrow{R_3-R_1+R_2} \quad
\begin{bmatrix}
1 & 1 & -3 \\
0 & 2 & 1 \\
0 & 0 & 0
\end{bmatrix}
= U,
\]

we conclude that \( \{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \} \) is a basis for \( \text{row}(A) \).

Q: In \( A \rightarrow U \), why is it sufficient to reduce \( A \) only to row echelon form \( U \)?

A: As the remark following Example 3.46 explains and then demonstrates by example,
the nonzero rows of \( U \) are linearly independent. That is all that is required. Why?

We should also note that provided \( A \rightarrow U \) uses no row interchanges,
the corresponding rows in \( A \) are also linearly independent.

Whence, it is obvious that those rows form a basis for \( \text{row}(A) \).

**col}(A): A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent.

When \( A \rightarrow U \), the columns with leading entries in \( U \) are linearly independent.

As in Example 3.47, the corresponding columns in \( A \) are also linearly independent.

Whence, it is obvious that those columns form a basis for \( \text{col}(A) \).

\[
A = \begin{bmatrix}
1 & 1 & -3 \\
0 & 2 & 1 \\
1 & -1 & -4
\end{bmatrix}
\quad \rightarrow \quad
\begin{bmatrix}
1 & 1 & -3 \\
0 & 2 & 1 \\
0 & 0 & 0
\end{bmatrix}
= U,
\]

we conclude that \( \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \} \) is a basis for \( \text{col}(A) \).

**null}(A): Since \( Av = 0 \) implies \( v \) is in \( \text{null}(A) \), we solve \( \begin{bmatrix} A & 0 \end{bmatrix} \rightarrow \begin{bmatrix} U' & 0 \end{bmatrix} \) to find the conditions.

We row reduce \( U \) one more step to \( U' \) make it easier to find the conditions:

\[
A = \begin{bmatrix}
1 & 1 & -3 \\
0 & 2 & 1 \\
1 & -1 & -4
\end{bmatrix}
\quad \xrightarrow{R_3-R_1+R_2} \quad
\begin{bmatrix}
1 & 1 & -3 \\
0 & 2 & 1 \\
0 & 0 & 0
\end{bmatrix}
\quad \xrightarrow{R_1+3R_2} \quad
\begin{bmatrix}
1 & 7 & 0 \\
0 & 2 & 1 \\
0 & 0 & 0
\end{bmatrix}
= U'
\]

\[
\begin{bmatrix} U' & 0 \end{bmatrix} = \begin{bmatrix}
1 & 7 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \Rightarrow \quad
\begin{cases}
x_1 + 7x_2 = 0 & x_1 = -7s \\
x_2 + x_3 = 0 & x_2 = -2s
\end{cases}
\Rightarrow \quad
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
-2
\end{bmatrix}
\]

Since \( s \) is arbitrary, \( \text{null}(A) = \text{span} \left( \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix} \right) \).

So, \( \left\{ \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix} \right\} \) is a basis for \( \text{null}(A) \).
21. We find bases for \( \text{row}(A) \) and \( \text{col}(A) \) following Examples 3.45 and 3.47 respectively.

**row(\(A\)):** A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent. Clearly, the linearly independent *columns* of \( A^T \) do just that.

When \( A^T \rightarrow R \), the columns with leading 1s in \( R \) are linearly independent.

As in Example 3.47, the corresponding columns in \( A^T \) are also linearly independent.

Whence, it is obvious that the *transposes* of those columns form a basis for row(\(A\)).

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
-1 & 1 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} = R
\]

we conclude that \( \{ [1 \ 0 \ -1], [1 \ 1 \ 1] \} \) is a basis for row(\(A\)).

**col(\(A\)):** A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent.

When \( A^T \rightarrow R \), the linearly independent *rows* (the nonzero rows) of \( R \) do just that.

Whence, it is obvious that the *transposes* of those rows form a basis for \( \text{col}(A) \).

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
-1 & 1 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix} = R
\]

we conclude that \( \{ [1 \ 0 \ -1], [0 \ 1 \ 1] \} \) is a basis for \( \text{col}(A) \).

We should also note that provided \( A^T \rightarrow R \) uses no row interchanges, the corresponding rows in \( A^T \) are also linearly independent.

Whence, it is obvious that the *transposes* of those rows form a basis for \( \text{col}(A) \).

22. We find bases for \( \text{row}(A) \) and \( \text{col}(A) \) following Examples 3.45 and 3.47 respectively.

**row(\(A\)):** A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent. Clearly, the linearly independent *columns* of \( A^T \) do just that.

When \( A^T \rightarrow R \), the columns with leading 1s in \( R \) are linearly independent.

As in Example 3.47, the corresponding columns in \( A^T \) are also linearly independent.

Whence, it is obvious that the *transposes* of those columns form a basis for row(\(A\)).

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 2 & -1 \\
-3 & 1 & -4 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix} = R,
\]

we conclude that \( \{ [1 \ 1 \ -3], [0 \ 2 \ 1] \} \) is a basis for row(\(A\)).

**col(\(A\)):** A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent. When \( A^T \rightarrow R \), the linearly independent *rows* (the nonzero rows) of \( R \) do just that.

Whence, it is obvious that the *transposes* of those rows form a basis for \( \text{col}(A) \).

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 2 & -1 \\
-3 & 1 & -4 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix} = R,
\]

we conclude that \( \{ [1 \ 0 \ 1], [0 \ 1 \ -1] \} \) is a basis for \( \text{col}(A) \).
24. We find bases for \( \text{row}(A) \) and \( \text{col}(A) \) following Examples 3.45 and 3.47 respectively.

**row}(A):** A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent. Clearly, the linearly independent *columns* of \( A^T \) do just that. When \( A^T \rightarrow R \), the columns with leading 1s in \( R \) are linearly independent. As in Example 3.47, the corresponding columns in \( A^T \) are also linearly independent. Whence, it is obvious that the *transposes* of those columns form a basis for \( \text{row}(A) \).

Since \( A^T = \begin{bmatrix} 2 & -1 & 1 \\ -4 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R \),

we conclude that \( \{ \begin{bmatrix} 2 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \} \) is a basis for \( \text{row}(A) \).

**\( \text{col}(A) \):** A basis for \( \text{col}(A) \) must span the columns of \( A \) and be linearly independent. When \( A^T \rightarrow R \), the linearly independent *rows* (the nonzero rows) of \( R \) do just that. Whence, it is obvious that the *transposes* of those rows form a basis for \( \text{col}(A) \).

Since \( A^T = \begin{bmatrix} 2 & -1 & 1 \\ -4 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R \),

we conclude that \( \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \} \) is a basis for \( \text{col}(A) \).
39. If \( \text{nullity}(A) > 0 \), then the columns of \( A \) are linearly dependent. Though we could prove this using theorems, it is instructive to prove it directly.

If \( \text{nullity}(A) > 0 \), then there exists a vector \( x \neq 0 \) such that \( Ax = 0 \).

Let \( A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \) and \( x^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \).

Since \( x \neq 0 \) at least one \( c_i \neq 0 \). Then \( Ax = \sum c_i a_i = 0 \) where at least one \( c_i \neq 0 \).

Therefore, the columns of \( A \) are linearly dependent.

So all we have to show is: If \( A \) is a \( 3 \times 5 \) matrix, then \( \text{nullity}(A) > 0 \).

\[
\text{nullity}(A) = n - \text{rank}(A) = 5 - \text{rank}(A) \geq 5 - 3 = 2 > 0
\]

Q: If \( A \) is \( 3 \times 5 \), why is it obvious that \( \text{rank}(A) \leq 3 \)?

A: Recall, that the number of vectors in a basis for \( \text{row}(A) = \dim(\text{row}(A)) \).

Now note the rows contain a basis for \( \text{row}(A) \), so \( \dim(\text{row}(A)) \leq \) number of rows.

So, \( \text{rank}(A) = \dim(\text{row}(A)) \leq \) the number of rows = 3.

40. If the number of rows > the number of columns, then the rows are linearly dependent.

Why? Rank must satisfy the following two conditions simultaneously:

1) \( \text{rank}(A) = \dim(\text{row}(A)) \leq \) the number of rows
2) \( \text{rank}(A) = \dim(\text{col}(A)) \leq \) the number of columns

Therefore, rank must be less than or equal to the smaller of these two numbers.

So, \( \dim(\text{row}(A)) = \text{rank}(A) \leq \) the number of columns < the number of rows.

That is, \( \dim(\text{row}(A)) = \) the number of vectors in a basis for \( \text{row}(A) \) < the number of rows.

Recall, we can find a basis for \( \text{row}(A) \) from among the rows of \( A \).

Therefore, the fact that \( \dim(\text{row}(A)) < \) the number of rows implies the following:

There exists at least one row that is a linear combination of the remaining rows.

That is, the rows of \( A \) are linearly dependent.

If \( A \) is a \( 4 \times 2 \) matrix, then the rows are linearly dependent.

This is now obvious because the number of rows = \( 4 > 2 = \) the number of columns.

Q: When we compare this result to Theorem 2.8 of Section 2.3, what do we notice?

Q: If the number of columns > the number of rows, are the columns are linearly dependent?

41. Rank must satisfy the following two conditions simultaneously:

1) \( \text{rank}(A) = \dim(\text{row}(A)) \leq \) the number of rows
2) \( \text{rank}(A) = \dim(\text{col}(A)) \leq \) the number of columns

Therefore, rank must be less than or equal to the smaller of these two numbers.

Since \( A \) is \( 3 \times 5 \), \( \text{rank}(A) \) can equal 0, 1, 2, or 3.

Therefore, since \( n = 5 \) and \( \text{nullity}(A) = n - \text{rank}(A) \), we have:

\[
\text{nullity}(A) = 5 - 3 = 2, \quad 5 - 2 = 3, \quad 5 - 1 = 4, \quad \text{or} \quad 5 - 0 = 5
\]

42. Since \( A \) is \( 4 \times 2 \), \( \text{rank}(A) \) can equal 0, 1, or 2.

Therefore, since \( n = 2 \) and \( \text{nullity}(A) = n - \text{rank}(A) \), we have:

\[
\text{nullity}(A) = 2 - 2 = 0, \quad 2 - 1 = 1, \quad \text{or} \quad 2 - 0 = 2.
\]
43. \[ A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}_{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 2 & a \\ 0 & a + 1 & a + \frac{1}{2} \\ 0 & 0 & \frac{(a+1)(a-2)}{2} \end{bmatrix} \]

If \( a = -1 \), then \( A \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 1. \)

If \( a = 2 \), then \( A \rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2. \) Otherwise, \( \text{rank}(A) = 3. \)

44. \[ A = \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix}_{R_1 \leftrightarrow R_3} \begin{bmatrix} -2 & -1 & a \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix}_{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & a - 2 \\ 3 & 3 & -2 \\ a & 2 & -1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 2 & a - 2 \\ 0 & 1 & a - \frac{4}{3} \\ 0 & 2 & -2a - (a-1)^2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & a - 2 \\ 0 & 1 & a - \frac{4}{3} \\ 0 & 0 & (a-1)(a-\frac{5}{3}) \end{bmatrix} \]

If \( a = 1, \frac{5}{3} \), then \( A \rightarrow \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2. \) Otherwise, \( \text{rank}(A) = 3. \)

45. As in Example 3.52, \( \{u, v, w\} \) form a basis for \( \mathbb{R}^3 \). When \( A = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \), \( \text{rank}(A) = 3. \)

\[ A = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \]

Since \( \text{rank}(A) = 3 \), \( \{u, v, w\} \) form a basis for \( \mathbb{R}^3 \).

46. As in Example 3.52, \( \{u, v, w\} \) form a basis for \( \mathbb{R}^3 \). When \( A = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \), \( \text{rank}(A) = 3. \)

\[ A = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 5 & -3 \\ 3 & 1 & 1 \end{bmatrix}_{R_3 + R_1} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 4 & -3 \\ 3 & 1 & 1 \end{bmatrix}_{R_4 \rightarrow R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 4 & -2 \end{bmatrix} \]

Since \( \text{rank}(A) < 3 \), \( \{u, v, w\} \) does not form a basis for \( \mathbb{R}^3 \).
47. As in Example 3.52, \{x, u, v, w\} form a basis for \(\mathbb{R}^4 \iff \text{When } A = \begin{bmatrix} x^T \\ u^T \\ v^T \\ w^T \end{bmatrix}, \text{rank}(A) = 4.\]

\[
A = \begin{bmatrix}
    1 & 1 & 0 \\
    1 & 0 & 1 \\
    1 & 0 & 1 \\
    0 & 1 & 1 \\
\end{bmatrix} R_2 - R_1 \rightarrow \begin{bmatrix}
    1 & 1 & 1 & 0 \\
    0 & 0 & -1 & 1 \\
    0 & -1 & 0 & 1 \\
    0 & 1 & 1 & 1 \\
\end{bmatrix} R_3 + R_2 + R_4 \rightarrow \begin{bmatrix}
    1 & 1 & 1 & 0 \\
    0 & 0 & -1 & 1 \\
    0 & 0 & 0 & 3 \\
    0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Since \(\text{rank}(A) = 4\), \{x, u, v, w\} form a basis for \(\mathbb{R}^4\).

48. As in Example 3.52, \{x, u, v, w\} form a basis for \(\mathbb{R}^4 \iff \text{When } A = \begin{bmatrix} x^T \\ u^T \\ v^T \\ w^T \end{bmatrix}, \text{rank}(A) = 4.\]

\[
A = \begin{bmatrix}
    x^T \\
    u^T \\
    v^T \\
    w^T \\
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 & -1 \\
    -1 & 1 & 0 & 0 \\
    0 & 0 & -1 & 1 \\
    0 & -1 & 1 & 0 \\
\end{bmatrix} R_2 - R_4 \rightarrow \begin{bmatrix}
    1 & 0 & 0 & -1 \\
    0 & -1 & 1 & 0 \\
    0 & 0 & -1 & 1 \\
    -1 & 1 & 0 & 0 \\
\end{bmatrix} R_4 + R_2 + R_3 \rightarrow \begin{bmatrix}
    1 & 0 & 0 & -1 \\
    0 & -1 & 1 & 0 \\
    0 & 0 & -1 & 1 \\
    0 & 0 & 1 & -1 \\
\end{bmatrix}
\]

Since \(\text{rank}(A) = 4\), \{x, v, w\} form a basis for \(\mathbb{R}^4\).

49. We find the coordinate vector \([w]_B\) by finding \(c_1\) and \(c_1\) such that \(w = c_1 b_1 + c_1 b_2\).
As in Example 2.18 of Section 2.3, we form the matrix \(A = \begin{bmatrix} b_1 & b_2 & w \end{bmatrix}\) and row reduce.

\[
A = \begin{bmatrix} b_1 & b_2 & w \end{bmatrix} = \begin{bmatrix}
    1 & 1 & 1 \\
    2 & 0 & 6 \\
    0 & -1 & 2 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 0 & 3 \\
    0 & 1 & -2 \\
    0 & 0 & 0 \\
\end{bmatrix}
\]

Since \(w = 3b_1 - 2b_2\), we have the coordinate vector \([w]_B = \begin{bmatrix} 3 \\
    -2 \end{bmatrix}\).

50. We find the coordinate vector \([w]_B\) by finding \(c_1\) and \(c_1\) such that \(w = c_1 b_1 + c_1 b_2\).
As in Example 2.18 of Section 2.3, we form the matrix \(A = \begin{bmatrix} b_1 & b_2 & w \end{bmatrix}\) and row reduce.

\[
A = \begin{bmatrix} b_1 & b_2 & w \end{bmatrix} = \begin{bmatrix}
    3 & 5 & 1 \\
    1 & 1 & 3 \\
    4 & 6 & 4 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
    1 & 0 & 7 \\
    0 & 1 & -4 \\
    0 & 0 & 0 \\
\end{bmatrix}
\]

Since \(w = 7b_1 - 4b_2\), we have the coordinate vector \([w]_B = \begin{bmatrix} 7 \\
    -4 \end{bmatrix}\).

51. We row reduce over \(\mathbb{Z}_2\) to find \(\text{rank}(A)\), then \(\text{nullity}(A) = n - \text{rank}(A)\).

\[
A = \begin{bmatrix}
    1 & 1 & 0 \\
    0 & 1 & 1 \\
    1 & 0 & 1 \\
\end{bmatrix} R_3 + R_1 \rightarrow \begin{bmatrix}
    1 & 1 & 0 \\
    0 & 1 & 1 \\
    0 & 0 & 1 \\
\end{bmatrix}
\]

Since \(\text{rank}(A) = 2\), we have \(\text{nullity}(A) = 3 - 2 = 1\).