3.5 Subspaces, Basis, Dimension, and Rank

1. As in Example 3.38, substituting the condition $x = 0$ into $\begin{bmatrix} x \\ y \end{bmatrix}$ yields $\begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   Since y is arbitrary, $S = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. So, S is a subspace of $\mathbb{R}^2$ by Theorem 3.19.

   Q: Geometrically speaking, what is $x = 0$?
   A: The equation $x = 0$ is a line through the origin. This should come as no surprise. Why? Because a line through the origin is a subspace in $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$.

2. We should suspect that set S defined by $x \geq 0$ and $y \geq 0$ is not a subspace. Why? Because multiplication by scalars (e.g., negative numbers) remove values from the set S.

   How do we prove that S is *not* a subspace?
   We provide a counterexample to show that one of the required properties does *not* hold.

   So, note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in S, but $-1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is not.

   Property (3) ($\mathbf{u}$ in S implies $c\mathbf{u}$ in S) fails so S is not a subspace of $\mathbb{R}^2$.

3. As in Example 3.38, substituting the condition $y = 2x$ into $\begin{bmatrix} x \\ y \end{bmatrix}$ yields $\begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   Since x is arbitrary, $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$. So, S is a subspace of $\mathbb{R}^2$ by Theorem 3.19.

   Q: Geometrically speaking, what is $y = 2x$?
   A: Since $y = 2x \Rightarrow 2x - y = 0$, this is obviously a line through the origin.

4. We should suspect that set S defined by $xy \geq 0$ is not a subspace. Why? Because, in general, when an inequality is a condition it is hard to have closure.

   How do we prove that S is *not* a subspace?
   We provide a counterexample to show that one of the required properties does *not* hold.

   Note $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are in S, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not.

   Property (2) ($\mathbf{u}$, $\mathbf{v}$ in S implies $\mathbf{u}+\mathbf{v}$ in S) fails so S is not a subspace of $\mathbb{R}^2$.

5. As in Example 3.38, substituting the condition $x = y = z$ into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ yields $\begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   Since x is arbitrary, $S = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$. So, S is a subspace of $\mathbb{R}^3$ by Theorem 3.19.

   Q: Geometrically speaking, what is $x = y = z$?
   A: Since $x = y = z \Rightarrow x - y = 0$ and $x - z = 0$, this is obviously a line through the origin.
   Why? Because $x - y = 0$ and $x - z = 0$ are planes through the origin that intersect in that line.
   Note: a plane through the origin is a subspace in $\mathbb{R}^3, \mathbb{R}^4, \cdots, \mathbb{R}^n$.

6. As in Example 3.38, substituting $z = 2x$ and $y = 0$ into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ yields $\begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

   Since x is arbitrary, $S = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)$. So, S is a subspace of $\mathbb{R}^3$ by Theorem 3.19.

   Q: Geometrically speaking, what is $z = 2x$ and $y = 0$?
   A: Since $2x - z = 0$ and $y = 0$ are planes through the origin,
   their intersection is a line through the origin.

7. We should recognize right away that set S defined by $x - y + z = 1$ is *not* a subspace. Why?
   Because $x - y + z = 1$ is a plane that does *not* pass through the origin. So we conclude:

   Property (1) ($\mathbf{0}$ is in S) fails so S is not a subspace of $\mathbb{R}^3$.

   Q: How can we tell that $x - y + z = 1$? is a plane that does *not* pass through the origin?
   A: The equation of a plane through the origin must be of the form $ax + by + cz = 0$. Why?
   Because the zero vector must satisfy the equation. So, we must have: $a0 + b0 + c0 = 0$.

8. We should suspect that set S defined by $|x - y| = |y - z|$ is *not* a subspace. Why?
   Because, in general, when an absolute value is a condition, it is difficult to have closure.

   How do we prove that S is *not* a subspace?
   We provide a counterexample to show that one of the required properties does *not* hold.

   Note $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ are in S, but $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ is not.

   Property (2) ($\mathbf{u}$, $\mathbf{v}$ in S implies $\mathbf{u} + \mathbf{v}$ in S) fails so S is not a subspace of $\mathbb{R}^3$.

9. Every line ℓ through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$
   because if ℓ has equation $\mathbf{x} = \mathbf{0} + t\mathbf{d}$ then $\ell = \text{span}(\mathbf{d})$.
   Therefore, once again, Theorem 3.19 implies that ℓ is a subspace of $\mathbb{R}^3$.

   We might also have stated our proof this way:

ℓ is a line $\Rightarrow$	ℓ has equation $\mathbf{x} = \mathbf{p} + t\mathbf{d}$
ℓ passes through the origin $\Rightarrow$	$\mathbf{x} = \mathbf{0} + t\mathbf{d} = t\mathbf{d}$
The definition of span $\Rightarrow$	$\ell = \text{span}(\mathbf{d})$
So, Theorem 3.19 $\Rightarrow$	ℓ is a subspace of $\mathbb{R}^3$

   Q: Did we use the fact that ℓ passes through the origin?
   A: Yes, in Step 2. Otherwise $\mathbf{0}$ will not be in ℓ as required.

   Q: Did we use the fact that ℓ is a line in $\mathbb{R}^3$?
   A: No. The parametric equation is the same in $\mathbb{R}^n$, so our conclusion is true in $\mathbb{R}^n$.

10. Let S be the set of points in $\mathbb{R}^2$ that are on the x-axis or the y-axis (or both).
    Since $\mathbf{u}$, $\mathbf{v}$ in S does *not* imply $\mathbf{u} + \mathbf{v}$ is in S, S is not a subspace of $\mathbb{R}^2$.
    For example, $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$ are in S, but $\mathbf{u} + \mathbf{v} = (1, 1)$ is not.

11. As in Example 3.41, we will determine whether **b** is in col(A) and **w** is in row(A).

col(A): To say **b** is in col(A) means **b** is a linear combination of the columns of A. So:

$\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ if the system $A\mathbf{x} = \mathbf{b}$ is consistent.

We row reduce the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \Rightarrow$

The system is consistent. So, $\mathbf{b} \in \text{col}(A)$. In particular, $\mathbf{b} = 3\mathbf{a}_1 - \mathbf{a}_2$.

row(A): To say **w** is in row(A) means **w** is a linear combination of the rows of A. So:

$$\mathbf{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \in \text{row}(A) \text{ if } \left[\frac{A}{\mathbf{w}}\right] \longrightarrow \left[\frac{A}{\mathbf{0}}\right]$$

by elementary row operations *excluding* row interchanges involving the *last* row.

So, we have $\left[\frac{A}{\mathbf{w}}\right] = \left[\begin{array}{ccc} 1 & 0 & -1 \\ 1 & 1 & 1 \\ \hline -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-R_1} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \hline 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \\ \hline 0 & 0 & -2 \end{array}\right] \Rightarrow$

We cannot make the last row all zeroes $\Rightarrow \mathbf{w} \notin \text{row}(A)$.

12. As in Example 3.41, we will determine whether **b** is in col(A) and **w** is in row(A).

col(A): To say **b** is in col(A) means **b** is a linear combination of the columns of A. So:

$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$ if $A\mathbf{x} = \mathbf{b}$ is consistent.

We row reduce the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$

The system is consistent. So, $\mathbf{b} \in \text{col}(A)$. In particular, $\mathbf{b} = \frac{1}{2}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2(A)$.

row(A): To say **w** is in row(A) means **w** is a linear combination of the rows of A. So:

$$\mathbf{w} = \begin{bmatrix} 2 & 4 & -5 \end{bmatrix} \in \text{row}(A) \text{ if } \left[\frac{A}{\mathbf{w}}\right] \longrightarrow \left[\frac{A}{\mathbf{0}}\right]$$

by elementary row operations *excluding* row interchanges involving the *last* row.

So, $\left[\frac{A}{\mathbf{w}}\right] = \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \\ \hline 2 & 4 & -5 \end{array}\right] \xrightarrow[R_4-2R_1]{R_3-R_1} \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \\ \hline 0 & 2 & 1 \end{array}\right] \xrightarrow[R_4-R_2]{R_3+R_2} \left[\begin{array}{ccc} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \Rightarrow$

So, **w** is a linear combination of the rows of $A \Rightarrow \mathbf{w} \in \text{row}(A)$.