3.4 The *LU* Factorization

1. Since $A = LU$, where L is unit lower triangular and U is upper triangular, we have:

$$A\mathbf{x} = \mathbf{b} \overset{A=LU}{\Rightarrow} (LU)\mathbf{x} = \mathbf{b} \overset{(AB)C=A(BC)}{\Rightarrow} L(U\mathbf{x}) = \mathbf{b} \overset{U\mathbf{x}=\mathbf{y}}{\Rightarrow} L\mathbf{y} = \mathbf{b}$$

So we solve the system by the two-step method outlined after Theorem 3.15:
1) Solve $L\mathbf{y} = \mathbf{b}$ by *forward substitution* (see Exercises 25 and 26 in Section 2.1) and
2) Solve $U\mathbf{x} = \mathbf{y}$ by *back substitution* (see Example 2.5 in Section 2.1)

Since $A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix} = LU$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $L\mathbf{y} = \mathbf{b} \Rightarrow$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{r} y_1 \quad = 5 \\ -y_1 + y_2 = 1 \end{array} \Rightarrow \begin{array}{l} y_1 = 5 \\ y_2 = y_1 + 1 \end{array} \qquad \text{So, } \mathbf{y} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Likewise, since $U = \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $U\mathbf{x} = \mathbf{y} \Rightarrow$

$$\begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Rightarrow \begin{array}{r} -2x_1 + x_2 = 5 \\ 6x_2 = 6 \end{array} \Rightarrow \begin{array}{l} x_1 = \frac{1}{2}x_2 - \frac{5}{2} \\ x_2 = 1 \end{array} \qquad \text{So, } \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Check: $A\mathbf{x} = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \mathbf{b}$.

2. We solve the system by the two-step method outlined after Theorem 3.15:

$$A\mathbf{x} = \mathbf{b} \overset{A=LU}{\Rightarrow} (LU)\mathbf{x} = \mathbf{b} \overset{U\mathbf{x}=\mathbf{y}}{\Rightarrow} L\mathbf{y} = \mathbf{b}$$

Since $A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix} = LU$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$, $L\mathbf{y} = \mathbf{b} \Rightarrow$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \Rightarrow \begin{array}{r} y_1 \quad = 0 \\ \frac{1}{2}y_1 + y_2 = 8 \end{array} \Rightarrow \begin{array}{l} y_1 = 0 \\ y_2 = -\frac{1}{2}y_1 + 8 \end{array} \qquad \text{So, } \mathbf{y} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}.$$

Likewise, since $U = \begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$, $U\mathbf{x} = \mathbf{y} \Rightarrow$

$$\begin{bmatrix} 4 & -2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \Rightarrow \begin{array}{rcl} 4x_1 - 2x_2 &=& 0 \\ 4x_2 &=& 8 \end{array} \Rightarrow \begin{array}{rcl} x_1 &=& \frac{1}{2}x_2 \\ x_2 &=& 2 \end{array} \qquad \text{So, } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Check: $A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \mathbf{b}$.

3. Since $A = LU$, where L is unit lower triangular and U is upper triangular, we have:

$$A\mathbf{x} = \mathbf{b} \overset{A=LU}{\Rightarrow} (LU)\mathbf{x} = \mathbf{b} \overset{(AB)C=A(BC)}{\Rightarrow} L(U\mathbf{x}) = \mathbf{b} \overset{U\mathbf{x}=\mathbf{y}}{\Rightarrow} L\mathbf{y} = \mathbf{b}$$

So we solve the system by the two-step method outlined after Theorem 3.15:

1) Solve $L\mathbf{y} = \mathbf{b}$ by *forward substitution* (see Exercises 25 and 26 in Section 2.1) and

2) Solve $U\mathbf{x} = \mathbf{y}$ by *back substitution* (see Example 2.5 in Section 2.1)

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix} = LU \text{ and } \mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \text{ so } L\mathbf{y} = \mathbf{b} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{5}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} y_1 \qquad\qquad &= -3 \\ -y_1 + y_2 \qquad &= 1 \\ 2y_1 - \tfrac{5}{4}y_2 + y_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} y_1 &= -3 \\ y_2 &= y_1 + 1 \\ y_3 &= -2y_1 + \tfrac{5}{4}y_2 \end{aligned} \Rightarrow$$

$$\begin{aligned} y_1 &= -3 \\ y_2 &= -3 + 1 = -2 \\ y_3 &= -2(-3) + \tfrac{5}{4}(-2) = \tfrac{7}{2} \end{aligned} \quad \text{So, } \mathbf{y} = \begin{bmatrix} -3 \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

$$\text{Likewise, } U = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} -3 \\ -2 \\ \frac{7}{2} \end{bmatrix}, \text{ so } U\mathbf{x} = \mathbf{y} \Rightarrow$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & -\frac{7}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ \frac{7}{2} \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + x_2 - 2x_3 &= -3 \\ 4x_2 - 6x_3 &= -2 \\ -\tfrac{7}{2}x_3 &= \tfrac{7}{2} \end{aligned} \Rightarrow \begin{aligned} x_1 &= -\tfrac{1}{2}x_2 + x_3 - \tfrac{3}{2} \\ x_2 &= \tfrac{3}{2}x_3 - \tfrac{1}{2} \\ x_3 &= -1 \end{aligned} \Rightarrow$$

$$\begin{aligned} x_1 &= -\tfrac{1}{2}(-2) - 1 - \tfrac{3}{2} = -\tfrac{3}{2} \\ x_2 &= \tfrac{3}{2}(-1) - \tfrac{1}{2} = -2 \\ x_3 &= -1 \end{aligned} \Rightarrow \quad \text{So, } \mathbf{x} = \begin{bmatrix} -\frac{3}{2} \\ -2 \\ -1 \end{bmatrix}$$

$$\text{Check: } A\mathbf{x} = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 4 & -3 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \mathbf{b}.$$

6. We solve the system by the two-step method outlined after Theorem 1:

$$A\mathbf{x} = \mathbf{b} \overset{A=LU}{\Rightarrow} (LU)\mathbf{x} = \mathbf{b} \overset{U\mathbf{x}=\mathbf{y}}{\Rightarrow} L\mathbf{y} = \mathbf{b}$$

$$L\mathbf{y} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -6 \\ -3 \end{bmatrix} \Rightarrow$$

$$U\mathbf{x} = \mathbf{y} \Rightarrow \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -6 \\ -3 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{16}{3} \\ -\frac{10}{3} \\ 3 \\ -3 \end{bmatrix}$$

$$\text{Check: } A\mathbf{x} = \begin{bmatrix} 1 & 4 & 3 & 0 \\ -2 & -5 & -1 & 2 \\ 3 & 6 & -3 & -4 \\ -5 & -8 & 9 & 9 \end{bmatrix} \begin{bmatrix} \frac{16}{3} \\ -\frac{10}{3} \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix} = \mathbf{b}$$

7. Following the *multiplier* method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} \xrightarrow{R_2'=R_2+3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = U \Rightarrow l_{21} = -3 \Rightarrow L = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = A$$

8. Following the *multiplier* method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix} \xrightarrow{R_2'=R_2-\frac{3}{2}R_1} \begin{bmatrix} 2 & -4 \\ 0 & 7 \end{bmatrix} = U \Rightarrow l_{21} = \frac{3}{2} \Rightarrow L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix} = A$$

9. Following the *multiplier* method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 7 & 9 \end{bmatrix} \xrightarrow[R_3'=R_2-8R_1]{R_2'=R_2-4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -9 & -15 \end{bmatrix} \xrightarrow{R_3''=R_3'-3R_2'} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 3 \end{bmatrix} = U \Rightarrow$$

$$\begin{matrix} l_{21} = 4 \\ l_{31} = 8 \quad l_{32} = 3 \end{matrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 3 & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 7 & 9 \end{bmatrix} = A$$

10. Following the *multiplier* method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[R_3'=R_3-\frac{3}{2}R_1]{R_2'=R_2-2R_1} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 1 & \frac{11}{2} \end{bmatrix} \xrightarrow{R_3''=R_3'+\frac{1}{4}R_2'} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} = U \Rightarrow$$

$$\begin{aligned} l_{21} &= 2 \\ l_{31} &= \tfrac{3}{2} \quad l_{32} = -\tfrac{1}{4} \end{aligned} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{3}{2} & -\frac{1}{4} & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{3}{2} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix}$$

11. Following the *multiplier* method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & -6 & 7 \\ -1 & -2 & -9 & 0 \end{bmatrix} \xrightarrow[R_4'=R_4+R_1]{R_2'=R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 4 & -3 & 2 \\ 0 & 6 & -6 & 7 \\ 0 & 0 & -6 & -1 \end{bmatrix} \xrightarrow{R_3'=R_3-\frac{3}{2}R_2'} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 4 & -3 & 2 \\ 0 & 0 & -\frac{3}{2} & 4 \\ 0 & 0 & -6 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4''=R_4'-4R_3'} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 4 & -3 & 2 \\ 0 & 0 & -\frac{3}{2} & 4 \\ 0 & 0 & 0 & -17 \end{bmatrix} = U \Rightarrow \begin{aligned} l_{21} &= 2 \\ l_{31} &= 0 \quad l_{32} = \tfrac{3}{2} \\ l_{41} &= -1 \quad l_{42} = 0 \quad l_{43} = 4 \end{aligned} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ -1 & 0 & 4 & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ -1 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 4 & -3 & 2 \\ 0 & 0 & -\frac{3}{2} & 4 \\ 0 & 0 & 0 & -17 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & -6 & 7 \\ -1 & -2 & -9 & 0 \end{bmatrix}$$

12. Following the *multiplier* method of Example 3.35, we find the LU factorization of A:

$$A = \begin{bmatrix} 2 & 2 & 2 & 1 \\ -2 & 4 & -1 & 2 \\ 4 & 4 & 7 & 3 \\ 6 & 9 & 5 & 8 \end{bmatrix} \xrightarrow[\substack{R_3'=R_3-2R_1 \\ R_4'=R_4-3R_1}]{R_2'=R_2+R_1} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 3 & -1 & 5 \end{bmatrix} \xrightarrow{R_4''=R_4'-\frac{1}{2}R_2'} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

$$\xrightarrow{R_4'''=R_4''+\frac{1}{2}R_3'} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U \Rightarrow \begin{aligned} l_{21} &= -1 \\ l_{31} &= 2 \quad l_{32} = 0 \\ l_{41} &= 3 \quad l_{42} = \tfrac{1}{2} \quad l_{43} = -\tfrac{1}{2} \end{aligned} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 1 \\ -2 & 4 & -1 & 2 \\ 4 & 4 & 7 & 3 \\ 6 & 9 & 5 & 8 \end{bmatrix}$$

13. By adapting the *multiplier* method of Example 3.35, we find the LU factorization of A. However, in this case, we simply note that A is already upper triangular so $L = I_3$.

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Q: Why is $L = I_3$, the 3×3 identity matrix instead of I_4, the 4×4 identity matrix?
A: Because A has 3 rows. Considering only size, we have $A = LU = [3 \times 3][3 \times 4] = [3 \times 4]$.

14. By adapting the *multiplier* method of Example 3.35, we find the LU factorization of A:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -7 & 3 & 8 & -2 \\ 1 & 1 & 3 & 5 & 2 \\ 0 & 3 & -3 & -6 & 0 \end{bmatrix} \xrightarrow[R_3'=R_3-R_1]{R_2'=R_2+2R_1} \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & -3 & 3 & 6 & 0 \\ 0 & -1 & 3 & 6 & 1 \\ 0 & 3 & -3 & -6 & 0 \end{bmatrix} \xrightarrow[R_4''=R_4+R_2']{R_3''=R_3'-\frac{1}{3}R_2'} \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & -3 & 3 & 6 & 0 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\begin{aligned} & l_{21} = -2 \\ \Rightarrow\ & l_{31} = 1 \quad l_{32} = \tfrac{1}{3} \\ & l_{41} = 0 \quad l_{42} = -1 \quad l_{43} = 0 \end{aligned} \quad \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & \frac{1}{3} & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & \frac{1}{3} & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & -3 & 3 & 6 & 0 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -7 & 3 & 8 & -2 \\ 1 & 1 & 3 & 5 & 2 \\ 0 & 3 & -3 & -6 & 0 \end{bmatrix}$$

15. Since $A = LU \Rightarrow A^{-1} = U^{-1}L^{-1}$, we use the LU factorization of A to find A^{-1}:

Since $A = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix} = LU$, we have:

$$L^{-1} = \tfrac{1}{1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } U^{-1} = -\tfrac{1}{12}\begin{bmatrix} 6 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{bmatrix}.$$

$$\text{Therefore, } A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{12} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$\text{Check: } AA^{-1} = \begin{bmatrix} -2 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -\frac{5}{12} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Note: To compute L^{-1} and U^{-1}, we used the formula: $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

18. We compute A^{-1} one column at a time using the method outlined below:

$$(LU)\mathbf{x}_i = \mathbf{e}_i \overset{U\mathbf{x}_i=\mathbf{y}_i}{\Rightarrow} L\mathbf{y}_i = \mathbf{e}_i$$

We begin by computing column 1 of A^{-1}:

$$L\mathbf{y}_1 = \mathbf{e}_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{y}_1 = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \Rightarrow$$

$$U\mathbf{x}_1 = \mathbf{y}_1 \Rightarrow \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \Rightarrow$$

We repeat this process to compute column 2 of A^{-1}:

$$L\mathbf{y}_2 = \mathbf{e}_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$$

$$U\mathbf{x}_2 = \mathbf{y}_2 \Rightarrow \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix} \Rightarrow$$

We repeat one last time to compute column 3 of A^{-1}:

$$L\mathbf{y}_3 = \mathbf{e}_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$U\mathbf{x}_3 = \mathbf{y}_3 \Rightarrow \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{x}_3 = \begin{bmatrix} -\frac{4}{5} \\ -\frac{2}{5} \\ \frac{1}{2} \end{bmatrix} \Rightarrow$$

Therefore, $A^{-1} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{2}{5} & -\frac{4}{5} \\ -\frac{1}{2} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}$ exactly as we found in Exercise 16.

19. Since we need $R_1 \to R_2$, $R_2 \to R_3$, and $R_3 \to R_1$ one possibility is:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = E_{13}E_{12} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Q: Is this factorization of P or could we have found another one?
A: No, it is not unique. For instance, we could have chosen $P = E_{12}E_{23}$.

20. $P = E_{12}E_{14}E_{13}$, that is: $\begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}\begin{bmatrix} 0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0 \end{bmatrix}\begin{bmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{bmatrix}.$

21. We find one possible factorization by tracing the row interchanges: $P = E_{13}E_{23}E_{34}$.

22. One possibility is: $P = E_{12}E_{45}E_{24}E_{34}$.

23. To find the P^TLU factorization of A, we begin by permuting the rows of A:

Since $A = \begin{bmatrix} 0&1&4\\-1&2&1\\1&3&3 \end{bmatrix}$, we have $PA = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}\begin{bmatrix} 0&1&4\\-1&2&1\\1&3&3 \end{bmatrix} = \begin{bmatrix} -1&2&1\\0&1&4\\1&3&3 \end{bmatrix}.$

Now we follow the *multiplier* method of Example 3.35 to find the LU factorization of PA:

$$PA = \begin{bmatrix} -1&2&1\\0&1&4\\1&3&3 \end{bmatrix} \xrightarrow{R_3'=R_3+R_1} \begin{bmatrix} -1&2&1\\0&1&4\\0&5&4 \end{bmatrix} \xrightarrow{R_3''=R_3'-5R_2} \begin{bmatrix} -1&2&1\\0&1&4\\0&0&-16 \end{bmatrix} = U \Rightarrow$$

$$\begin{matrix} l_{21} = 0 \\ l_{31} = -1 \quad 0 \quad l_{32} = 5 \end{matrix} \Rightarrow L = \begin{bmatrix} 1&0&0\\0&1&0\\-1&5&1 \end{bmatrix}$$

Check: $LU = \begin{bmatrix} 1&0&0\\0&1&0\\-1&5&1 \end{bmatrix}\begin{bmatrix} -1&2&1\\0&1&4\\0&0&-16 \end{bmatrix} = \begin{bmatrix} -1&2&1\\0&1&4\\1&3&3 \end{bmatrix} = PA$

Now $PA = LU \Rightarrow A = P^{-1}LU$, but $P^{-1} = P^T$, so we have $A = P^TLU$.

Check: $P^TLU = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0\\0&1&0\\-1&5&1 \end{bmatrix}\begin{bmatrix} -1&2&1\\0&1&4\\0&0&-16 \end{bmatrix} = \begin{bmatrix} 0&1&4\\-1&2&1\\1&3&3 \end{bmatrix} = A$

25. To find the P^TLU factorization of A, we begin by permuting the rows of A:

$$\text{Since } A = \begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \text{ we have } PA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Now we follow the *multiplier* method of Example 3.35 to find the LU factorization of PA:

$$PA = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_4' = R_4 + R_2} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U \Rightarrow$$

$$\begin{matrix} l_{21} = 0 & & \\ l_{31} = 0 & l_{32} = 0 & \\ l_{41} = 0 & l_{42} = -1 & l_{43} = 0 \end{matrix} \quad \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\text{Check: } LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} = PA$$

Now $PA = LU \Rightarrow A = P^{-1}LU$, but $P^{-1} = P^T$, so we have $A = P^TLU$.

$$\text{Check: } P^TLU = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = A$$

26. As we construct the permutation matrix P, we will count the row interchange choices.

For the first row interchange, there are n choices.
That is, we can interchange row 1 with any of the n rows of P.

For the second row interchange, there are $n-1$ choices.
That is, we can interchange row 2 with any of the remaining $n-1$ rows of P.

So, for the first two row interchanges we have $n(n-1)$ choices.
Continuing in this manner, we see there are $n(n-1)\cdots 2 \cdot 1 = n!$ choices.

That is, there are $n!$ permutation matrices of size $n \times n$.