3.3 The Inverse of a Matrix

1. As is Example 3.24, we begin by computing the determinant of A, $\det A$. Why? Since if $\det A = 0$, then A is not invertible.

   Since $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$, $\det A = ad - bc = 4(2) - 1(7) = 1$, A is invertible.

   So, by Theorem 3.8, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$.

   Q: How can we check our answer for A^{-1}?

   A: By applying the definition. That is, by checking that $AA^{-1} = I$. Let's do that:

   $$AA^{-1} = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 4(2)+7(-1) & 1(2)+2(-1) \\ 4(-7)+7(4) & 1(-7)+2(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

   Q: Should $A^{-1}A = I$, too?

   A: Yes. We should check that as well. On the calculator, these are quick and simple to do.

2. As is Example 3.24, we begin by computing the determinant of A, $\det A$. Why?

   Since $A = \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix}$, $\det A = 4(0) - (-2)(2) = 4$, A is invertible.

   So, by Theorem 3.8, $A^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$.

   Check: $AA^{-1} = \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ or $A^{-1}A = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix} = I$.

3. As is Example 3.24, we begin by computing the determinant of A, $\det A$. Why?

   Since $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, $\det A = 3(8) - 4(6) = 0$, A is not invertible.

4. Since $\det A = 0(0) - (-1)(1) = 1$, A is invertible. So, by Theorem 3.8, we have:

   $$A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

   Check: $AA^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ or $A^{-1}A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = I$.

5. Since $\det A = \frac{3}{4}(\frac{2}{3}) - \frac{3}{5}(\frac{5}{6}) = 0$, A is not invertible.

6. Since $\det A = (\frac{\sqrt{2}}{2})(\sqrt{2}) - (-\sqrt{2})(2\sqrt{2}) = 5$, A is invertible. So, by Theorem 3.8:

$$A^{-1} = \frac{1}{5}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -2\sqrt{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{5} & \frac{\sqrt{2}}{5} \\ -\frac{2\sqrt{2}}{5} & \frac{\sqrt{2}}{10} \end{bmatrix}.$$

Check: $AA^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2} \\ 2\sqrt{2} & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{5} & \frac{\sqrt{2}}{5} \\ -\frac{2\sqrt{2}}{5} & \frac{\sqrt{2}}{10} \end{bmatrix} = \begin{bmatrix} \frac{2}{10}+\frac{8}{10} & -\frac{2}{5}+\frac{2}{5} \\ -\frac{8}{10}+\frac{8}{10} & \frac{8}{10}+\frac{2}{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

7. Since $\det A = (-1.5)(2.4) - (-4.2)(0.5) = -1.5$, A is invertible. So, by Theorem 3.8:

$$A^{-1} = -\frac{1}{1.5}\begin{bmatrix} 2.4 & 4.2 \\ -0.5 & -1.5 \end{bmatrix} = \begin{bmatrix} -1.6 & -2.8 \\ 0.\overline{3} & 1 \end{bmatrix}$$

Check: $AA^{-1} = \begin{bmatrix} -1.5 & -4.2 \\ 0.5 & 2.4 \end{bmatrix}\begin{bmatrix} -1.6 & -2.8 \\ 0.\overline{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

8. Since $\det A = (2.54)(0.8) - (8.128)(0.25) = 0$, A is not invertible.

9. Since $\det A = (a)(a) - (-b)(b) = a^2 + b^2$, provided a and b are not both zero, A is invertible.

$$A^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix} \quad (a \text{ and } b \text{ not both zero})$$

Check: $AA^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix} = \begin{bmatrix} \frac{a^2+b^2}{a^2+b^2} & \frac{ab-ba}{a^2+b^2} \\ \frac{ba-ab}{a^2+b^2} & \frac{b^2+a^2}{a^2+b^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

10. Since $\det A = \frac{1}{a}\cdot\frac{1}{d} - \frac{1}{b}\cdot\frac{1}{c} = \frac{bc-ad}{abcd}$, when a, b, c, d not zero, $bc \neq ad$, A is invertible.

$$A^{-1} = \frac{abcd}{bc-ad}\begin{bmatrix} \frac{1}{d} & -\frac{1}{b} \\ -\frac{1}{c} & \frac{1}{a} \end{bmatrix} = \begin{bmatrix} \frac{abc}{bc-ad} & \frac{acd}{ad-bc} \\ \frac{abd}{ad-bc} & \frac{bcd}{bc-ad} \end{bmatrix} \quad (a, b, c, d \text{ all not zero and } bc \neq ad)$$

Check: $AA^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{d} \end{bmatrix}\begin{bmatrix} \frac{abc}{bc-ad} & \frac{acd}{ad-bc} \\ \frac{abd}{ad-bc} & \frac{bcd}{bc-ad} \end{bmatrix} = \begin{bmatrix} \frac{bc-ad}{bc-ad} & \frac{-cd+cd}{ad-bc} \\ \frac{ab-ab}{ad-bc} & \frac{-ad+bc}{bc-ad} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

11. As in Example 3.25, we use the inverse of the coefficient matrix A to solve the system. That is, the reasoning we will employ here is: $A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$.

Since $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, we have $A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

Therefore, since $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, we have $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$.

Q: How can we check our answer for $\mathbf{x}$?

A: By applying the condition. That is, by checking that $A\mathbf{x} = \mathbf{b}$. Let's do that:

Check: $A\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \mathbf{b}.$

12. As in Example 3.25, we use the inverse of the coefficient matrix A to solve the system.

Since $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, we have $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

Therefore, since $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we have $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Check: $A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{b}$.

13. As in Example 3.25, we use the inverse of the coefficient matrix A to solve the systems.

Since $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$, we have $A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}$.

Also, note that we are given $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(a) Since $A\mathbf{x}_i = \mathbf{b}_i$, the solution in each case is $\mathbf{x}_i = A^{-1}\mathbf{b}_i$. So we have:

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1 = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{x}_2 = A^{-1}\mathbf{b}_2 = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_3 = A^{-1}\mathbf{b}_3 = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

(b) We form the augmented matrix $\left[\, A \mid \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \,\right]$ and row reduce to solve.

$$\left[\begin{array}{cc|ccc} 1 & 2 & 3 & -1 & 2 \\ 2 & 6 & 5 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 4 & -5 & 6 \\ 0 & 1 & -\frac{1}{2} & 2 & -2 \end{array}\right].$$

(c) The A^{-1} method requires 7 multiplications, while row reduction requires only 6.

14. To prove X is the inverse of A, all we have to show is $AX = I$.
Theorem 3.9b. asserts $(cA)^{-1} = \frac{1}{c}A^{-1}$, so all we need to show is $(cA)\left(\frac{1}{c}A^{-1}\right) = I$.

$$(cA)\left(\tfrac{1}{c}A^{-1}\right) = \left(\tfrac{1}{c}c\right)\left(AA^{-1}\right) = AA^{-1} = I$$

15. To prove X is the inverse of A, all we have to show is $AX = I$.
Theorem 3.9d. asserts $(A^T)^{-1} = (A^{-1})^T$, so all we need to show is $(A^T)(A^{-1})^T = I$.

$$(A^T)(A^{-1})^T \overset{\substack{\text{by Thm 3.4d} \\ B^TA^T=(AB)^T}}{=} (A^{-1}A)^T = I^T \overset{\substack{\text{This is obvious.} \\ \text{Why?}}}{=} I$$

19. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Then

$$\begin{aligned} (A+B)^{-1} &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \\ A^{-1} + B^{-1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

So $(A+B)^{-1} \neq A^{-1} + B^{-1}$.

20. $XA^2 = A^{-1} \Rightarrow (XA^2)(A^2)^{-1} = A^{-1}(A^2)^{-1} \Rightarrow X\left((A^2)(A^2)^{-1}\right) = A^{-1}(A^2)^{-1}$
So $X = A^{-1}(A^{-1})^2 = (A^{-1})^3 = A^{-3}$.

21. $AXB = (BA)^2 \Rightarrow A^{-1}(AXB)B^{-1} = A^{-1}(BA)^2B^{-1} \Rightarrow (A^{-1}A)X(BB^{-1}) = A^{-1}(BA)^2B^{-1}$
So $X = A^{-1}(BA)^2B^{-1}$.

22. $(A^{-1}X)^{-1} = A(B^{-2}A)^{-1} \Rightarrow X^{-1}(A^{-1})^{-1} = A(A^{-1}(B^{-2})^{-1}) \Rightarrow X^{-1}A = B^2$
So $X = (B^2A^{-1})^{-1} = (A^{-1})^{-1}(B^2)^{-1} = AB^{-2}$.

23. $ABXA^{-1}B^{-1} = I + A \Rightarrow X = B^{-1}A^{-1}(I+A)BA = B^{-1}A^{-1}BA + B^{-1}A^{-1}ABA$
So $X = (AB)^{-1}BA + A$.

24. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

25. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

26. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

27. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

28. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

29. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

30. The row operations $R_2 - 2R_1 - 2R_3$, $R_3 + R_1$, will turn matrix A into matrix D. So,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \text{ satisfies } EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix} = D.$$

However, E is not an elementary matrix since it incorporates two elementary row operations.

31. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$.

32. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

34. $\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$.

35. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

36. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

37. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

38. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

39. As in Example 3.29, we attempt to express A as a product of elementary matrices. Why? To compute A^{-1} and to *factor* A, that is to see how A is created by elementary row operations.

As we row reduce A, we use the elementary operations involved to create E_i.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We recreate those steps on the identity matrix I to create E_1 and E_2.
This calculation can be done mentally, but we write it here to demonstrate the process.

Since $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, we get $E_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Since $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, we get $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

Since $(E_2E_1)A = I$, by definition $A^{-1} = E_2E_1$.
So, Theorem 3.9a implies $A = (A^{-1})^{-1} = (E_2E_1)^{-1} = E_1^{-1}E_2^{-1}$.

But E_1^{-1} and E_2^{-1} can be written down without doing any calculation. How?
Since E_1 was created by $R_2 + R_1$, we create E_1^{-1} by $R_2 - R_1$.

Likewise, since E_2 was created by $-\frac{1}{2}R_2$, we create E_2^{-1} by $-2R_2$.

Therefore, $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

We could verify this by performing the indication operation on the identity matrix I.
Or we could simply verify the claim that $A = E_1^{-1}E_2^{-1}$ directly.

Check: $A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = E_1^{-1}E_2^{-1}$

This answer is not unique. For example, we could have row reduced A as follows.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \xrightarrow{R_2-\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$ and $E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$.

Verify that $A = E_1^{-1}E_2^{-1}$ in this case as well.

40. As in Example 3.29, we attempt to express A as a product of elementary matrices. Why? To compute A^{-1} and to *factor* A, that is to see how A is created by elementary row operations.

As we row reduce A, we use the elementary operations involved to create E_i and E_i^{-1}.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, and $E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

And $E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and $E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Since $E_4E_3E_2E_1A = I$, we should have $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$.

$$\text{Check: } A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

This answer is not unique. For example, we could have row reduced A as follows.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

And $E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $E_4^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Verify that $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$ in this case as well.

41. Suppose $AB = I$. Then, consider: $B\mathbf{x} = \mathbf{0}$. Left-multiplying by $A \Rightarrow AB\mathbf{x} = A\mathbf{0}$.
This implies that $I\mathbf{x} = \mathbf{x} = \mathbf{0}$. Thus $B\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0} \Rightarrow$
The equivalence of (c) and (a) in the Fundamental Theorem (3.12) $\Rightarrow B$ is invertible.
If we now right-multiply both sides of $AB = I$ by B^{-1}, we obtain
$ABB^{-1} = IB^{-1} \Leftrightarrow AI = B^{-1} \Leftrightarrow A = B^{-1} \Leftrightarrow A^{-1} = B$.
Thus the inverse of A exists and $A^{-1} = B$.

42. (a) Let A be invertible and $AB = 0$. Now left-multiply both sides of $AB = 0$ by $A^{-1} \Rightarrow$ $A^{-1}AB = A^{-1}0 \Leftrightarrow IB = 0 \Leftrightarrow B = 0$.

(b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $B \neq 0$.

43. (a) Suppose that A is invertible (thus A^{-1} exists) and $BA = CA$.
If we now right-multiply both sides of $BA = CA$ by A^{-1}, we obtain
$BAA^{-1} = CAA^{-1} \Leftrightarrow BI = CI \Leftrightarrow B = C$.

(b) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = CA. \text{ So } BA = CA, \text{ but } B \neq C.$$

44. (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A,$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = B, \text{ and}$$

$$C^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = C.$$

So A, B, and C are idempotent 2×2 matrices.

(b) Let A be an invertible idempotent $n \times n$ matrix. Since A is idempotent we have $A^2 =$
But, A is invertible, so if we now right-multiply both sides of $AA = A$ by A^{-1}, we g

$$AAA^{-1} = AA^{-1} \Leftrightarrow AI = I \Leftrightarrow A = I.$$

Thus the only $n \times n$ invertible idempotent matrix is the identity matrix.

45. To prove X is the inverse of A, all we have to show is $AX = I$.
We claim $A^{-1} = 2I - A$, so all we need to show is $A(2I - A) = I$.

$$A^2 - 2A + I = O \overset{\text{given}}{\Rightarrow} 2A - A^2 = I \overset{\text{factor}}{\Rightarrow} A(2I - A) = I$$

46. Let A be an invertible symmetric matrix. Since A is invertible we have $AA^{-1} = I$.
If we now take the transpose of both sides of $AA^{-1} = I$, we obtain:

$$(AA^{-1})^T = I^T \Rightarrow (A^{-1})^T A^T = I \Rightarrow (A^{-1})^T A = I \Rightarrow (A^{-1})^T AA^{-1} = IA^{-1} \Rightarrow$$
$$(A^{-1})^T I = A^{-1} \Rightarrow (A^{-1})^T = A^{-1}.$$

Thus A^{-1} is equal to its own transpose and so, by definition, is symmetric.
We conclude that the inverse of a symmetric matrix is symmetric.

47. Let A and B be square matrices and assume that AB is invertible.

Then $AB(AB)^{-1} = A\left(B(AB)^{-1}\right) = I$, showing that A is invertible with $A^{-1} = B(AB)^{-}$

Therefore $B(AB)^{-1}A = I$, showing that B is invertible with $B^{-1} = (AB)^{-1}A$.

48. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A \,|\, I\,]$ to $[\,I \,|\, A^{-1}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$.

Q: How does $[\,A \,|\, I\,] \longrightarrow [\,I \,|\, A^{-1}\,]$ relate to our understanding that $A^{-1} = E_3E_2E_1$?
A: Multiplication by the E_i performs the associated operations on the identity matrix.

49. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A \,|\, I\,]$ to $[\,I \,|\, A^{-1}$

$$[\,A \,|\, I\,] = \left[\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 1 & \frac{3}{10} & \frac{1}{5} \end{array}\right] = [\,I \,|\, A^{-1}\,] \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

50. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A\,|\,I\,]$ to $[\,I\,|\,A^{-1}\,]$.

$$[\,A\,|\,I\,] = \left[\begin{array}{rr|rr} 6 & -4 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 6 & -4 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{array}\right] \neq [\,I\,|\,A^{-1}\,] \Rightarrow$$

Since the left matrix cannot be reduced to I (why?), we conclude A^{-1} does not exist.

51. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A\,|\,I\,]$ to $[\,I\,|\,A^{-1}\,]$.

$$[\,A\,|\,I\,] = \left[\begin{array}{rr|rr} 1 & a & 1 & 0 \\ -a & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & 0 & \frac{1}{a^2+1} & -\frac{a}{a^2+1} \\ 0 & 1 & \frac{a}{a^2+1} & \frac{1}{a^2+1} \end{array}\right] = [\,I\,|\,A^{-1}\,] \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{a^2+1} & -\frac{a}{a^2+1} \\ \frac{a}{a^2+1} & \frac{1}{a^2+1} \end{bmatrix}.$$

Q: What is the restriction on a?
A: Since $a^2+1=0 \Rightarrow a^2=-1$, there is no restriction on a. Why not?

Q: Does this agree with the formula for A^{-1} given in Theorem 3.8, that is $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$?

52. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A\,|\,I\,]$ to $[\,I\,|\,A^{-1}\,]$.

$$[\,A\,|\,I\,] = \left[\begin{array}{rrr|rrr} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$

53. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A\,|\,I\,]$ to $[\,I\,|\,A^{-1}\,]$.

$$[\,A\,|\,I\,] = \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 5 & -5 & -2 & 0 & 1 \end{array}\right] \neq [\,I\,|\,A^{-1}\,] \Rightarrow$$

Since the left matrix cannot be reduced to I (why?), we conclude A^{-1} does not exist.

54. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A\,|\,I\,]$ to $[\,I\,|\,A^{-1}\,]$.

$$[\,A\,|\,I\,] = \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] = [\,I\,|\,A^{-1}\,]$$

55. As in Example 3.30, we adjoin the identity matrix to A then row reduce $[\,A\,|\,I\,]$ to $[\,I\,|\,A^{-1}\,]$.

$$[\,A\,|\,I\,] = \left[\begin{array}{rrr|rrr} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a^2} & \frac{1}{a} & 0 \\ 0 & 0 & 1 & \frac{1}{a^3} & -\frac{1}{a^2} & \frac{1}{a} \end{array}\right] = [\,I\,|\,A^{-1}\,]$$

Q: The entries of A^{-1} imply $a \neq 0$. Why is that obvious in the original matrix A?

A: If $a=0$ in A, we have $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, which is obviously not invertible. Why?

64. $\begin{bmatrix} A & B \\ O & D \end{bmatrix} \begin{bmatrix} A & B \\ O & D \end{bmatrix}^{-1} = \begin{bmatrix} A & B \\ O & D \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix} = \begin{bmatrix} AA^{-1}+BO & -AA^{-1}BD^{-1} \\ OA^{-1}+DO & -OA^{-1}BD^{-1} \end{bmatrix}$

$= \begin{bmatrix} I & -BD^{-1}+BD^{-1} \\ O & I \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}.$

65. $\begin{bmatrix} O & B \\ C & I \end{bmatrix} \begin{bmatrix} O & B \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} O & B \\ C & I \end{bmatrix} \begin{bmatrix} -(BC)^{-1} & (BC)^{-1}B \\ C(BC)^{-1} & I - C(BC)^{-1}B \end{bmatrix}$

$= \begin{bmatrix} BC(BC)^{-1} & B - BC(BC)^{-1}B \\ -C(BC)^{-1} + C(BC)^{-1} & C(BC)^{-1}B + I - C(BC)^{-1}B \end{bmatrix}$

$= \begin{bmatrix} I & B-B \\ O & I \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}.$

66. $\begin{bmatrix} I & B \\ C & I \end{bmatrix} \begin{bmatrix} I & B \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} I & B \\ C & I \end{bmatrix} \begin{bmatrix} (I-BC)^{-1} & -(I-BC)^{-1}B \\ -C(I-BC)^{-1} & I + C(I-BC)^{-1}B \end{bmatrix}$

$= \begin{bmatrix} (I-BC)^{-1} - BC(I-BC)^{-1} & -(I-BC)^{-1}B + BI + BC(I-BC)^{-1}B \\ C(I-BC)^{-1} - C(I-BC)^{-1} & -C(I-BC)^{-1}B + I + C(I-BC)^{-1}B \end{bmatrix}$

$= \begin{bmatrix} (I-BC)(I-BC)^{-1} & (-I+BC)(I-BC)^{-1}B - B \\ O & I \end{bmatrix}$

$= \begin{bmatrix} I & -B+B \\ O & I \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}.$

67. $\begin{bmatrix} O & B \\ C & D \end{bmatrix} \begin{bmatrix} O & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} O & B \\ C & D \end{bmatrix} \begin{bmatrix} -\left(BD^{-1}C\right)^{-1} & \left(BD^{-1}C\right)^{-1}BD^{-1} \\ D^{-1}C\left(BD^{-1}C\right)^{-1} & D^{-1} - D^{-1}C\left(BD^{-1}C\right)^{-1}BD^{-} \end{bmatrix}$

$= \begin{bmatrix} BD^{-1}C\left(BD^{-1}C\right)^{-1} & BD^{-1} - BD^{-1}C\left(BD^{-1}C\right)^{-1}BD^{-1} \\ -C\left(BD^{-1}C\right)^{-1} + DD^{-1}C\left(BD^{-1}C\right)^{-1} & C\left(BD^{-1}C\right)^{-1}BD^{-1} + DD^{-1} + \left(-DD^{-1}C\left(BD^{-1}C\right)^{-1}BD\right. \end{bmatrix}$

$= \begin{bmatrix} I & O \\ O & I \end{bmatrix}.$

71. We partition the matrix $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ into the form $\begin{bmatrix} O & B \\ C & I \end{bmatrix}$ as in Exercise 65. Then

$$-(BC)^{-1} = -\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right)^{-1} = -\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(BC)^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$C(BC)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \text{ and}$$

$$I - C(BC)^{-1}B = I - \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$

72. We partition the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ -1 & 5 & 2 \end{bmatrix}$ into the form $\begin{bmatrix} O & B \\ C & D \end{bmatrix}$ as in Exercise 67. Then

$$-(BD^{-1}C)^{-1} = -\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)^{-1} = -([-5])^{-1} = \left[\tfrac{1}{5}\right],$$

$$(BD^{-1}C)^{-1}BD^{-1} = -\tfrac{1}{5}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \end{bmatrix},$$

$$D^{-1}C(BD^{-1}C)^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\left(-\tfrac{1}{5}\right) = \begin{bmatrix} -\frac{3}{5} \\ \frac{8}{5} \end{bmatrix}, \text{ and}$$

$$D^{-1} - D^{-1}C(BD^{-1}C)^{-1}BD^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} - \begin{bmatrix} -\frac{3}{5} \\ \frac{8}{5} \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} \end{bmatrix}.$$

Thus, $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ -1 & 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{8}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix}.$