hw11

3.2 Matrix Algebra

1. Following remarks prior to Example 3.16, the key assumption is matrices are the same size. Then add, subtract, and multiply (by scalars only) as in *normal* algebra.

$$X - 2A + 3B = 0 \Rightarrow X = 2A - 3B = \begin{bmatrix} 5 & 4 \\ 3 & 5 \end{bmatrix}.$$

2. Following remarks prior to Example 3.16, the key assumption is matrices are the same size. Then add, subtract, and multiply (by scalars only) as in *normal* algebra.

$$2X = A - B \Rightarrow X = \tfrac{1}{2}(A - B) = \begin{bmatrix} 1 & 1 \\ 1 & \frac{3}{2} \end{bmatrix}.$$

3. $X = \frac{2}{3}(A + 2B) = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} \\ \frac{10}{3} & 4 \end{bmatrix}.$

4. $X = 5A - 2B = \begin{bmatrix} 7 & 10 \\ 13 & 18 \end{bmatrix}.$

5. As in Example 3.16, we want to find scalars c_1 and c_2 such that $c_1A_1 + c_2A_2 = B$.

$$c_1\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

The left-hand side of this equation can be rewritten as $\begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix}$

Comparing entries and the definition of matrix equality yields
$$\begin{array}{rcrcl} c_1 & & & = & 2 \\ 2c_1 & + & c_2 & = & 5 \\ -c_1 & + & 2c_2 & = & 0 \\ c_1 & + & c_2 & = & 3 \end{array}$$

Gauss-Jordan elimination easily gives
$$\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 2 & 1 & 5 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, $c_1 = 2$ and $c_2 = 1$. Thus, $2A_1 + A_2 = B$, which can be easily checked.

Having walked through the process, we note this pattern in our augmented matrix: the first column is the entries of A_1, the second column is the entries of A_2, and the third column, the augmented column, is the entries of B. Make use of this pattern!

13. Following Example 3.18, we create an augmented matrix and row reduce to solve.
As in Exercise 8, the first column is the entries of A_1, the second column is the entries of A_2, but now the augmented column is all zeroes.

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Clearly, the only solution is $c_1 = c_2 = 0$. What do we conclude?
We conclude that A_1 and A_2 are linearly independent.

14. Following Example 3.18, we create an augmented matrix and row reduce to solve.
As in Exercise 8, the first column is the entries of A_1, the second column is the entries of A_2, the third column is the entries of A_3, but now the augmented column is all zeroes.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $c_1 = -3c_3$, $c_2 = c_3$ is a solution with at least one $c_i \neq 0$. What does that tell us?
That tells us that A_1, A_2, and A_3 are linearly dependent.
In particular, if we let $c_3 = -1$, we have the following dependence relation:

$$3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

23. We will compute AB and BA, then equate entries to find the conditions on a, b, c, and d.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = BA$$

Equating entries gives us the following four equations (conditions on a, b, c, and d):
$a + c = a$, $b + d = a + b$, $c = c$, and $d = c + d \Rightarrow$ The conditions are $a = d$ and $c = 0$.

24. We will compute AB and BA, then equate entries to find the conditions on a, b, c, and d.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ -a+c & -b+d \end{bmatrix} = \begin{bmatrix} a-b & -a+b \\ c-d & -c+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Equating entries gives us the following four equations (conditions on a, b, c, and d):
$a - c = a - b$, $b - d = -a + b$, $-a + c = c - d$, and $-b + d = -c + d$.
So, the conditions on a, b, c, and d are $a = d$ and $b = c$.

25. We will compute AB and BA, then equate entries to find the conditions on a, b, c, and d.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = BA$$

Equating entries gives us the following four equations (conditions on a, b, c, and d):
$a + 2c = a + 3b$, $b + 2d = 2a + 4b$, $3a + 4c = c + 3d$, and $3b + 4d = 2c + 4d$.
So, the conditions on a, b, c, and d are $3b = 2c$ and $a = d - c$.

26. We will compute A_1B and BA_1, then equate entries to find conditions on a, b, c, and d. We will then repeat the process for A_4B and BA_4, then combine conditions for our answer.

$$A_1B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = BA_1$$

Equating entries gives us the following four equations (conditions on a, b, c, and d):
$a = a$, $b = 0$, $c = 0$, and $0 = 0 \Rightarrow$ The A_1 conditions are $b = 0$ and $c = 0$.

Repeating the process for A_4B and BA_4 yields:

$$A_4B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = BA_4$$

Equating entries gives us the following four equations (conditions on a, b, c, and d):
$0 = 0$, $b = 0$, $c = 0$, and $d = d \Rightarrow$ The A_4 conditions are $b = 0$ and $c = 0$.

Combining the conditions for A_1 and A_4 (in this case they are the same) gives us:
The required conditions so that B will commute with A_1 and A_4 are $b = c = 0$.

Q: Let $M = aA_1 + dA_4$. Does the B we found above commute with M?
A: Yes, since $BM = B(aA_1 + dA_4) = aBA_1 + dBA_4 = aA_1B + dA_4B = (aA_1 + dA_4)B = MB$.

36. (a) Let A be the $n \times n$ matrix with entries $a_{ij} = 1$ if either $i = 1$ or $j = 1$,, 0 otherwise, and let B be the $n \times n$ matrix with entries $b_{ij} = 1$ if $i + j = n + 1$, 0 otherwise.

$$\text{So, } A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Clearly, A and B are symmetric, but AB is not symmetric.

When $n = 2$, for instance, we have $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) Since this is an *if and only if* statement, we have two claims to prove.

if: If A, B, and AB are symmetric, then $AB = BA$.

$$AB \overset{\substack{AB \text{ is} \\ \text{symmetric}}}{=} (AB)^T \overset{\substack{\text{by} \\ \text{Thm 3.4d}}}{=} B^T A^T \overset{\substack{A \text{ and } B \text{ are} \\ \text{symmetric}}}{=} BA$$

only if: If A, B are symmetric and $AB = BA$, then AB is symmetric, that is $(AB)^T = AB$.

$$(AB)^T \overset{\substack{\text{by} \\ \text{Thm 3.4d}}}{=} B^T A^T \overset{\substack{A \text{ and } B \text{ are} \\ \text{symmetric}}}{=} BA \overset{\substack{\text{by the given} \\ AB = BA}}{=} AB$$

37. For each matrix, we will simply check to see if $A^T = -A$ is satisfied.

(a) Since $A^T = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \neq -\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = -A$, A is *not* skew-symmetric.

(b) Since $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$, A is skew-symmetric.

(c) Since $A^T = \begin{bmatrix} 0 & -3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} = -A$, A is skew-symmetric.

(d) Since $A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix} \neq -\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix} = -A$, A is *not* skew-symmetric.

38. A square matrix is called skew-symmetric if $A^T = -A \Leftrightarrow \left[A^T\right]_{ij} = [-A]_{ij} \Leftrightarrow [A]_{ji} = -[A]_{ij}$. Thus, the components must satisfy $a_{ji} = -a_{ij}$.

39. If A is skew-symmetric ($A^T = -A$), then the diagonal entries must be zero ($a_{ii} = 0$).

$$A^T = -A \quad \Rightarrow \quad \left[A^T\right]_{ij} = [-A]_{ij} \quad \Rightarrow \quad [A]_{ji} = -[A]_{ij}$$

$$\text{So } a_{ji} = -a_{ij} \quad \Rightarrow \quad a_{ii} = -a_{ii} \quad \Rightarrow \quad 2a_{ii} = 0 \quad \Rightarrow \quad a_{ii} = 0$$

40. If A and B are skew-symmetric, then so is $A + B$, that is $(A + B)^T = -(A + B)$.

$$(A+B)^T \overset{\substack{\text{by} \\ \text{Thm 3.4b}}}{=} A^T + B^T \overset{\substack{A \text{ and } B \text{ are} \\ \text{skew-symmetric}}}{=} (-A) + (-B) \overset{\substack{\text{by} \\ \text{Thm 3.2}}}{=} -(A+B)$$

1. Let A and B be skew-symmetric 2×2 matrices, so $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$.

Demanding that AB be skew-symmetric gives us the equation:

$$(AB)^T = -AB \Leftrightarrow \left(\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \right)^T = - \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix}^T = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} \Leftrightarrow \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix}.$$

So, $-ab = ab \Leftrightarrow ab = 0$. Letting $O =$ the zero matrix, we get:
AB will be skew-symmetric provided either $A = O$ or $B = O$ (or both).

42. If A is a square matrix, then $A - A^T$ is skew-symmetric, that is $(A - A^T)^T = -(A - A^T)$.

$$(A - A^T)^T \overset{\substack{\text{by} \\ \text{Thm 3.4b}}}{=} A^T - (A^T)^T \overset{\substack{\text{by} \\ \text{Thm 3.4a}}}{=} A^T - A \overset{\substack{\text{by} \\ \text{Thm 3.2}}}{=} -(A - A^T)$$

43. We will prove this claim in (a) and demonstrate it with an example in (b).

(a) If A is $n \times n$, then $A = B + C$, where B is symmetric and C is skew-symmetric.

$$S \overset{\substack{\text{symmetric by} \\ \text{Thm 3.5a}}}{=} A + A^T \quad \text{and} \quad S' \overset{\substack{\text{skew-symmetric by} \\ \text{Exercise 42}}}{=} A - A^T$$

Now simply note $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}S + \frac{1}{2}S'$

Q: When S is symmetric and S' is skew-symmetric, are cS and cS' also?
A: Yes, since $(cS)^T = cS^T = cS$ and $(cS')^T = c(S')^T = -cS'$.

(b)
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \frac{1}{2} \left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \right) + \frac{1}{2} \left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

44. Let A and B be $n \times n$ matrices, and let k be a scalar. Then

(i) $\operatorname{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn})$
$= (a_{11} + a_{22} + \cdots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn}) = tr(A) + tr(B)$.

(ii) $\operatorname{tr}(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = k \operatorname{tr}(A)$.

45. Let A and B be $n \times n$ matrices. Then

$$\begin{aligned} \operatorname{tr}(AB) &= (a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) + (a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}) + \\ &\qquad \cdots + (a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}) \\ &= (b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1}) + (b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2n}a_{n2}) + \\ &\qquad \cdots + (b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}) \\ &= \operatorname{tr}(BA) \end{aligned}$$

46. Let A be an $n \times n$ matrix. Then
$\operatorname{tr}(AA^T) = (a_{11}^2 + a_{12}^2 + \cdots + a_{1n}^2) + (a_{21}^2 + a_{22}^2 + \cdots + a_{2n}^2) + \cdots + (a_{n1}^2 + a_{n2}^2 + \cdots + a_{nn}^2)$,
that is, the sum of the squares of the entries of A.