HW10

1-7, 8, 34, 35 EZ 37

Matrix Operations

1. Following Examples 3.1 through 3.5, we have:

$$A+2D = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} + 2\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3+2(0) & 0+2(-3) \\ -1+2(-2) & 5+2(1) \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}.$$

2. Following Examples 3.1 through 3.5, we have:

$$3D-2A = 3\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3(0)-2(3) & 3(-3)-2(0) \\ 3(-2)-2(-1) & 3(1)-2(5) \end{bmatrix} = \begin{bmatrix} -6 & -9 \\ -4 & -7 \end{bmatrix}.$$

3. $B-C$ is not possible. Why not? B is a 2×3 matrix and C is a 3×2 matrix.
We can only add and subtract matrices of the same size.

4. Applying the definition the transpose to C^T, the transpose of matrix C, we have:

Since $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $B - C^T = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -5 & -4 \\ -2 & -2 & -3 \end{bmatrix}$.

5. By the definition of the matrix product, $C = AB$, and Example 3.6, we have: $AB =$

$$\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3(4)+0(0) & 3(-2)+0(2) & 3(1)+0(3) \\ (-1)(4)+5(0) & 3(-2)+(-1)(2) & 3(1)+(-1)(3) \end{bmatrix} = \begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}.$$

6. From the remarks following the definition of matrix multiplication, we see BD is not possible.
Why not? B is a 2×3 matrix and D is a 2×2 matrix. What does that tell us?
The number of columns in $B = 3 \neq 2 =$ the number of rows in D.
For matrix multiplication, the number of columns in B has to equal the number of rows in D.

In the future, when checking whether or not matrix multiplication is possible, we will write:
B is $[2 \times 3]$ and D is $[2 \times 2]$, so BD is $[2 \times 3][2 \times 2]$ which is not possible because the *inner* numbers do not match.

Q: Do they outer numbers have to match? If not, what do they tell us?
A: No, they do not have to match. The *outer* numbers tell us the size, $[r \times c]$, of the result.

7. We begin by applying the definition of matrix multiplication to see if BC is possible:
Since B is $[2 \times 3]$ and C is $[3 \times 2]$, BC, $[2 \times 3][3 \times 2]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $B = 3 =$ the number of rows in C.

Furthermore, since BC is $[2 \times 3][3 \times 2]$, BC will be a 2×2 matrix.
Since D is also a 2×2 matrix, we can add them together. That is, $D + BC$ is possible.

$$\text{First, } BC = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 4(1)-2(3)+1(5) & 4(2)-2(4)+1(6) \\ 0(1)+2(3)+3(5) & 0(2)+2(4)+3(6) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix}.$$

$$\text{So, } D+BC = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 21 & 26 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}.$$

8. We should note that B^TB is *always* possible. Why?
the number of columns of B^T = the number of rows of B (by the definition of B^T).

Since $B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$ and $B^T = \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix}$,

$$B^TB = \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4(4)+0(0) & 4(-2)+0(2) & 4(1)+0(3) \\ (-2)(4)+2(0) & (-2)(-2)+2(2) & (-2)(1)+2(3) \\ 1(4)+3(0) & 1(-2)+3(2) & 1(1)+3(3) \end{bmatrix}$$

$= \begin{bmatrix} 16 & -8 & 4 \\ -8 & 8 & 4 \\ 4 & 4 & 10 \end{bmatrix}$. Is BB^T always possible as well? Why or why not?

9. Before we begin, we should determine if AF and $E(AF)$ are possible.

Since A is $[2 \times 2]$ and F is $[2 \times 1]$, AF, $[2 \times 2][2 \times 1]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $A = 2 =$ the number of rows in F.
Furthermore, since AF is $[2 \times 2][2 \times 1]$, AF will be a 2×1 matrix.

Since E is $[1 \times 2]$ and AF is $[2 \times 1]$, $E(AF)$, $[1 \times 2][2 \times 1]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $E = 1 =$ the number of rows in AF.
Furthermore, since $E(AF)$ is $[1 \times 2][2 \times 1]$, $E(AF)$ will be a 1×1 matrix.

First, $AF = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(-1)+0(2) \\ (-1)(-1)+5(2) \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$.

So $E(AF) = [4 \ \ 2] \begin{bmatrix} -3 \\ 11 \end{bmatrix} = [4(-3)+2(11)] = [10]$.

10. Before we begin, we should determine if DF and $F(DF)$ are possible.

Since D is $[2 \times 2]$ and F is $[2 \times 1]$, DF, $[2 \times 2][2 \times 1]$, is possible. Why?
Because the *inner* numbers match. What does that tell us?
The number of columns in $D = 2 =$ the number of rows in F.
Furthermore, since DF is $[2 \times 2][2 \times 1]$, DF will be a 2×1 matrix.

Since F is $[2 \times 1]$ and DF is $[2 \times 1]$, $F(DF)$, $[2 \times 1][2 \times 1]$, is not possible. Why?
Because the *inner* numbers do not match. What does that tell us?
The number of columns in $F = 1 \neq 2 =$ the number of rows in DF.

So, $F(DF)$ is not possible.

19. The number of units of each product shipped to each warehouse is given by $A = \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix}$.

The cost of shipping one unit of each product is given by $B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$

(where b_{ij} is the cost of shipping a unit of product j by $i = 1$ truck, $i = 2$ train).
Compare the cost of shipping the products to each of the warehouses:

$$BA = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix} \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix} \Rightarrow$$

It is cheaper to ship the products to warehouse 1 by truck, but to warehouse 2 by train.

20. The costs of distributing one unit of product is given by $C = \begin{bmatrix} 0.75 & 0.75 & 0.75 \\ 1.00 & 1.00 & 1.00 \end{bmatrix}$

(where c_{ij} is the cost of shipping one unit of product j from warehouse i).
The cost of distribution is then given by:

$$CA^T = \begin{bmatrix} 0.75 & 0.75 & 0.75 \\ 1.00 & 1.00 & 1.00 \end{bmatrix} \begin{bmatrix} 200 & 75 \\ 150 & 100 \\ 100 & 125 \end{bmatrix} = \begin{bmatrix} 337.50 & 225.00 \\ 450.00 & 300.00 \end{bmatrix} \Rightarrow$$

It costs \$337.50 to ship all the products from warehouse 1, \$300.00 from warehouse 2.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

22. $\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

23. $A\mathbf{b}_1 = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix}$, $A\mathbf{b}_2 = 3\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ -4 \\ 0 \end{bmatrix}$,

and $A\mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ -4 \end{bmatrix}$. Therefore, $AB = \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix}$.

24. $\mathbf{A}_1B = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} - 2\begin{bmatrix} -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -9 & -8 \end{bmatrix}$,
$\mathbf{A}_2B = -3\begin{bmatrix} 2 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -4 & 5 \end{bmatrix}$, and

$\mathbf{A}_3B = 2\begin{bmatrix} 2 & 3 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -4 \end{bmatrix}$. Thus, $AB = \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix}$.

31. For matrices A, B we have the block structure $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \Rightarrow$

$$\begin{aligned}
AB &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 3 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.
\end{aligned}$$

32. $AB = \begin{bmatrix} 1 & 7 & 7 \\ -2 & 7 & 7 \end{bmatrix}.$

33. $AB = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 4 & 5 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$

34. $AB = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 2 & 3 & 6 & -1 \\ 3 & 3 & 4 & -2 \\ 4 & 4 & 4 & -4 \end{bmatrix}.$

35. (a) Computing the powers of matrix A as required, we have:

$$A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},\ A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

$$A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},\ A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = A!$$

(b) From our work in (*a*), we see that $A^1 = A^7 = A^{1 \cdot 6 + 1}$.
So the powers of A that actually create *distinct* matrices act like $\mathbb{Z}^6$.
See Section 1.4, Examples 1.32 through 1.35.
So, to determine A^{2001}, we should first determine the value of 2001 in $\mathbb{Z}^6$.
How? Divide 2001 by 6 and look at the remainder: $2001 = 333 \cdot 6 + 3 = 3$ in $\mathbb{Z}^6$.
Therefore $A^{2001} = A^{333 \cdot 6 + 3} = A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

36. As in Exercise 35, we compute the powers of B and look for patterns:

Since $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B^4 = \left(B^2\right)^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

So $B^8 = \left(B^4\right)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow B^9 = B^8 \cdot B = I \cdot B = B$.

Since $2001 = 250 \cdot 8 + 1 = 1$ in $\mathbb{Z}^8$, so $B^{2001} = B^{250 \cdot 8+1} = B^1 = B$. EC

37. Given $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we will prove $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for $n \geq 1$ using *induction.*

See Appendix B for discussion and examples of *Mathematical Induction.*

1: $A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This is obvious, so there is nothing to show.

n: $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. This is the induction hypothesis.

$n+1$: $A^{n+1} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$. This is the statement we must prove using the induction hypothesis.

$$A^{n+1} = A^1 A^n \overset{\text{by induction}}{=} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \overset{\text{by matrix multiplication}}{=} \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

We have proven (by induction) that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for $n \geq 1$.