5.4 Orthogonal Diagonalization of Symmetric Matrices

1. $\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3) = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 3 \Rightarrow D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

As a reminder, we show how to find $\mathbf{v}_1$ (make sure $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_1$).

$(A-5I)\mathbf{v}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis.

So, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$.

Verify $Q^T A Q = D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

2. $\begin{vmatrix} -1-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 2\lambda - 8 = 0 \Rightarrow \lambda_1 = -4, \lambda_2 = 2 \Rightarrow D = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$.

Furthermore, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

3. $\begin{vmatrix} 1-\lambda & \sqrt{2} \\ \sqrt{2} & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -1 \Rightarrow D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

Furthermore, $\mathbf{v}_1 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$.

4. $\begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 15\lambda + 50 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 10 \Rightarrow D = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

Furthermore, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$.

5. $\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(\lambda^2 - 2\lambda - 8) = 0 \Rightarrow D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

As a reminder, in this case we show how to find $\mathbf{v}_1$ ($\mathbf{v}_2$ and $\mathbf{v}_3$ are extremely similar).

$(A-5I)\mathbf{v}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 \textit{ is free} \\ x_2 = x_3 = 0 \end{matrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis.

So, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

6. Since $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

So, $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 4/5 & 3/5\sqrt{2} & 3/5\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -3/5 & 4/5\sqrt{2} & 4/5\sqrt{2} \end{bmatrix}$.

7. $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2 - 2\lambda) = 0 \Rightarrow D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$.

8. $\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 5)(\lambda + 1)^2 = 0 \Rightarrow D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

In this case, $(A - 5I)\mathbf{v} = \mathbf{0} \Rightarrow x_1 = \frac{1}{2}(x_2 + x_3)$. So, $x_2 = x_3 = 1 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$(A - (-1)I)\mathbf{v} = \mathbf{0}$ (of multiplicity 2) $\Rightarrow \begin{matrix} x_1 = -(x_2 + x_3) \\ \textit{or } x_2 = 1, x_3 = -1 \end{matrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

These choices make the $\mathbf{v}_i$ orthogonal. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}$.

9. $\begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2(\lambda - 2)^2 = 0 \Rightarrow D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$\lambda_1 = 2$ of multiplicity 2 $\Rightarrow (A - 2I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{matrix} x_2 = x_1 \\ x_4 = x_3 \end{matrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = 0$ of multiplicity 2 $\Rightarrow (A - 0I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{matrix} x_2 = -x_1 \\ x_4 = -x_3 \end{matrix} \Rightarrow \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

These choices make the $\mathbf{v}_i$ orthogonal. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$.

10. $\begin{vmatrix} 2-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-3)(\lambda-1)^3 = 0 \Rightarrow D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

Multiplicity 1 $\Rightarrow (A-3I)\mathbf{v} = \mathbf{0} \Rightarrow x_4 = x_1 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

Multiplicity 3 $\Rightarrow (A-1I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{matrix} x_4 = -x_1 \\ x_2,\ x_3\ free \end{matrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$

These choices make the $\mathbf{v}_i$ orthogonal. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{bmatrix}.$

11. $\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)^2 - b^2 = 0 \Rightarrow \lambda = a \pm b, \Rightarrow D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}.$

We show how to find $\mathbf{v}_1$ (make sure $\mathbf{v}_2$ is orthogonal to $\mathbf{v}_1$).

$(A-(a+b)I)\mathbf{v}_1 = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis.

So, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$

Verify $Q^TAQ = D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}.$

12. $\begin{vmatrix} a-\lambda & 0 & b \\ 0 & a-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-a)[(a-\lambda)^2 - b^2] = 0 \Rightarrow D = \begin{bmatrix} a+b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix}.$

As a reminder, in this case we show how to find $\mathbf{v}_1$ ($\mathbf{v}_2$ and $\mathbf{v}_3$ are extremely similar).

$(A-(a+b)I)\mathbf{v}_1 = \begin{bmatrix} -b & 0 & b \\ 0 & -b & 0 \\ b & 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = x_1 \\ x_2 = 0 \end{matrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

So, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Normalizing $\Rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}.$

13. (a) A, B orthogonally diagonalizable $\Rightarrow$ A, B symmetric $\Rightarrow$ $A+B$ symmetric $\Rightarrow$ $Q^T(A+B)Q = D$.

(b) A orthogonally diagonalizable $\Rightarrow$ A symmetric $\Rightarrow$ cA symmetric $\Rightarrow$ $Q^T(cA)Q = D$.

(c) AA^T is symmetric $\Rightarrow$ if A symmetric, then $AA^T = AA = A^2$ symmetric.
So, A orthogonally diagonalizable $\Rightarrow$ A symmetric $\Rightarrow$ A^2 symmetric $\Rightarrow$ $Q^T(A^2)Q = D$.

14. A invertible $\Rightarrow$ $Q^TAQ = D$ invertible $\Rightarrow$ $Q^TA^{-1}Q = D^{-1}$.

15. By Exercise 36 in Section 3.2, A, B symmetric and $AB = BA \Rightarrow AB$ symmetric.
So, A, B symmetric and $AB = BA$ (given) $\Rightarrow AB$ symmetric $\Rightarrow Q^T(AB)Q = D$.

16. Note, D diagonal $\Rightarrow B = QDQ^T$ symmetric since $(QDQ^T)^T = (Q^T)^TD^TQ^T = QDQ^T$.

$$A \text{ symmetric with } \lambda_i \geq 0 \Leftrightarrow Q^TAQ = \begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}^2 \Leftrightarrow Q^TAQ = D^2.$$

A symmetric with $\lambda_i \geq 0 \Rightarrow Q^TAQ = D^2 \Rightarrow A = QD^2Q^T = (QDQ^T)(QDQ^T) = B^2$.
$A = B^2 \Rightarrow Q^TAQ = Q^TB^2Q = (Q^TBQ)(Q^TBQ) = D^2 \Rightarrow A$ symmetric with $\lambda_i \geq 0$.

17. $A = \lambda_1\mathbf{q}_1\mathbf{q}_1{}^T + \lambda_2\mathbf{q}_2\mathbf{q}_2{}^T$

$$= 5\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + 3\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 5/2 & 5/2 \\ 5/2 & 5/2 \end{bmatrix} + \begin{bmatrix} 3/2 & -3/2 \\ -3/2 & 3/2 \end{bmatrix}.$$

18. $$A = -4\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} + 2\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}.$$

19. $$A = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

20. $$A = 5\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + (-1)\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 1/6 & 1/6 \\ -1/3 & 1/6 & 1/6 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 & 5/3 & 5/3 \\ 5/3 & 5/3 & 5/3 \\ 5/3 & 5/3 & 5/3 \end{bmatrix} + \begin{bmatrix} -2/3 & 1/3 & 1/3 \\ 1/3 & -1/6 & -1/6 \\ 1/3 & -1/6 & -1/6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}.$$

21. $$A = \lambda_1\mathbf{q}_1\mathbf{q}_1{}^T + \lambda_2\mathbf{q}_2\mathbf{q}_2{}^T = (-1)\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + 2\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -3/2 \\ -3/2 & 1/2 \end{bmatrix}.$$

22. $$A = 3\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} + (-3)\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} = \begin{bmatrix} -9/5 & 12/5 \\ 12/5 & 9/5 \end{bmatrix}.$$

23. $A = 1\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix} + 3\begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$

$= \begin{bmatrix} 5/3 & -2/3 & -1/3 \\ -2/3 & 5/3 & 1/3 \\ -1/3 & 1/3 & 8/3 \end{bmatrix}.$

24. $A = 0A + (-4)\begin{bmatrix} 1/3 & -1/3 & -1/3 \\ -1/3 & 1/3 & 1/3 \\ -1/3 & 1/3 & 1/3 \end{bmatrix} + (-4)\begin{bmatrix} 2/7 & -1/7 & 3/7 \\ -1/7 & 1/14 & -3/14 \\ 3/7 & -3/14 & 9/14 \end{bmatrix}$

$= \begin{bmatrix} -52/21 & 40/21 & -8/21 \\ 40/21 & -34/21 & -10/21 \\ -8/21 & -10/21 & -82/21 \end{bmatrix}.$

25. $\text{proj}_W(\mathbf{v}) = (\frac{\mathbf{q}\cdot\mathbf{v}}{\mathbf{q}\cdot\mathbf{q}})\mathbf{q} = (\mathbf{q}^T\mathbf{v})\mathbf{q} = \mathbf{q}(\mathbf{q}^T\mathbf{v}) = (\mathbf{q}\mathbf{q}^T)\mathbf{v}.$

Note: $(\mathbf{q}^T\mathbf{v})\mathbf{q} = \mathbf{q}(\mathbf{q}^T\mathbf{v})$ because $\mathbf{q}^T\mathbf{v}$ is a scalar.

26. (a) This follows from induction on Exercise 25.

(b) $P^T = \left(\mathbf{q}_1\mathbf{q}_1{}^T + \ldots + \mathbf{q}_k\mathbf{q}_k{}^T\right)^T = \mathbf{q}_1{}^T\mathbf{q}_1 + \ldots + \mathbf{q}_k{}^T\mathbf{q}_k = \mathbf{q}_1\mathbf{q}_1{}^T + \ldots + \mathbf{q}_k\mathbf{q}_k{}^T = P.$

$P^2 = (\mathbf{q}_1\mathbf{q}_1{}^T)(\mathbf{q}_1\mathbf{q}_1{}^T) + \ldots + (\mathbf{q}_k\mathbf{q}_k{}^T)(\mathbf{q}_k\mathbf{q}_k{}^T)$ (because $\mathbf{q}_i \cdot \mathbf{q}_j = 0$ for $i \neq j$)

$= \mathbf{q}_1(\mathbf{q}_1{}^T\mathbf{q}_1)\mathbf{q}_1{}^T + \ldots + \mathbf{q}_k(\mathbf{q}_k{}^T\mathbf{q}_k)\mathbf{q}_k = \mathbf{q}_1\mathbf{q}_1{}^T + \ldots + \mathbf{q}_k\mathbf{q}_k{}^T = P.$

(c) $QQ^T = \begin{bmatrix} \mathbf{q}_1 & \ldots & \mathbf{q}_k \end{bmatrix}\begin{bmatrix} \mathbf{q}_1{}^T \\ \vdots \\ \mathbf{q}_k{}^T \end{bmatrix} = \mathbf{q}_1\mathbf{q}_1{}^T + \ldots + \mathbf{q}_k\mathbf{q}_k{}^T = P.$

So, $\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$ orthonormal basis $\Rightarrow$
$\text{rank}(P) = \text{rank}(QQ^T) = \text{rank}(Q) = k$ by Section 3.4, Theorem 3.28.