5.3 The Gram-Schmidt Process and the QR Factorization

1. Applying Gram-Schmidt $\Rightarrow \mathbf{v}_1 = \mathbf{x}_1$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$.

   $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{v}_2\| = \frac{\sqrt{2}}{2} \Rightarrow \mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_2 = \frac{2}{\sqrt{2}}\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

2. Applying Gram-Schmidt $\Rightarrow \mathbf{v}_1 = \mathbf{x}_1$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{6}{18}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

   $\|\mathbf{v}_1\| = 3\sqrt{2}$, $\|\mathbf{v}_2\| = 2\sqrt{2} \Rightarrow \mathbf{q}_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_2 = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

3. G-S $\Rightarrow \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

   Therefore, $\|\mathbf{v}_1\| = \sqrt{3}$, $\|\mathbf{v}_2\| = \sqrt{6}$, $\|\mathbf{v}_3\| = \sqrt{2} \Rightarrow$

   $$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

4. $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$.

   Therefore, $\|\mathbf{v}_1\| = \sqrt{3}$, $\|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}$, $\|\mathbf{v}_3\| = \frac{\sqrt{2}}{2} \Rightarrow$

   $$\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}.$$

5. $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix} \Rightarrow$ Orthogonal basis $= \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix}\right\}$.

6. $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} - \frac{15}{10}\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ 1 \end{bmatrix} \Rightarrow$ Orthogonal basis $= \left\{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ -3/2 \\ 1 \end{bmatrix}\right\}$.

7. $\left(0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{9}\begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix}\right) + \left(\begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix} - \left(0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{4}{9}\begin{bmatrix} -1/2 \\ 1/2 \\ 2 \end{bmatrix}\right)\right) = \begin{bmatrix} -2/9 \\ 2/9 \\ 8/9 \end{bmatrix} + \begin{bmatrix} 38/9 \\ -38/9 \\ 19/9 \end{bmatrix}$.

8. $\left(\frac{1}{5}\begin{bmatrix}2\\-1\\1\\2\end{bmatrix}+\frac{8}{7}\begin{bmatrix}0\\-1/2\\-3/2\\1\end{bmatrix}\right)+\left(\begin{bmatrix}1\\4\\0\\2\end{bmatrix}-\left(\frac{1}{5}\begin{bmatrix}2\\-1\\1\\2\end{bmatrix}+\frac{8}{7}\begin{bmatrix}0\\-1/2\\-3/2\\1\end{bmatrix}\right)\right)=$

$$\begin{bmatrix}2/5\\-27/35\\-53/35\\54/35\end{bmatrix}+\begin{bmatrix}3/5\\167/35\\53/35\\16/35\end{bmatrix}.$$

9. $\mathbf{v}_2=\begin{bmatrix}1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\-1/2\\1/2\end{bmatrix}$, $\mathbf{v}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}-\frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1\\-1/2\\1/2\end{bmatrix}=\begin{bmatrix}2/3\\2/3\\-2/3\end{bmatrix}$.

Therefore, $\left\{\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1/2\\1/2\end{bmatrix},\begin{bmatrix}2/3\\2/3\\-2/3\end{bmatrix}\right\}$ is an orthogonal basis for col(A).

10. $\mathbf{v}_2=\begin{bmatrix}1\\-1\\1\\5\end{bmatrix}-\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\-2\\2\\4\end{bmatrix}$, $\mathbf{v}_3=\begin{bmatrix}1\\2\\0\\1\end{bmatrix}-\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}-0\begin{bmatrix}0\\-2\\2\\4\end{bmatrix}=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$.

11. $\mathbf{v}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}-\frac{1}{35}\begin{bmatrix}3\\1\\5\end{bmatrix}=\begin{bmatrix}-3/35\\34/35\\-1/7\end{bmatrix}$, $\mathbf{v}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}-\frac{1}{7}\begin{bmatrix}3\\1\\5\end{bmatrix}+\frac{5}{34}\begin{bmatrix}-3/35\\34/35\\-1/7\end{bmatrix}=\begin{bmatrix}-15/34\\0\\9/34\end{bmatrix}$.

Therefore, $\left\{\begin{bmatrix}3\\1\\5\end{bmatrix},\begin{bmatrix}-3/35\\34/35\\-1/7\end{bmatrix},\begin{bmatrix}-15/34\\0\\9/34\end{bmatrix}\right\}$ is an orthogonal basis.

12. $\mathbf{v}_2=\begin{bmatrix}1\\0\\3\\2\end{bmatrix}-0\begin{bmatrix}2\\1\\0\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\3\\2\end{bmatrix}$, $\mathbf{v}_3=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}-\frac{1}{6}\begin{bmatrix}2\\1\\0\\-1\end{bmatrix}-0\begin{bmatrix}1\\0\\3\\2\end{bmatrix}=\begin{bmatrix}-1/3\\5/6\\0\\1/6\end{bmatrix}$,

$$\mathbf{v}_4=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}-0\begin{bmatrix}2\\1\\0\\-1\end{bmatrix}-\frac{3}{14}\begin{bmatrix}1\\0\\3\\2\end{bmatrix}-0\begin{bmatrix}-1/3\\5/6\\0\\1/6\end{bmatrix}=\begin{bmatrix}-3/14\\0\\5/14\\-3/7\end{bmatrix}.$$

13. We need to choose an $\mathbf{x}_3$ and then use the Gram-Schmidt Process.

Since $\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$, we can let $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Why?

Q: How can we see that this is a good choice for $\mathbf{x}_3$?
A: It may help to temporarily ignore the fact that the columns of Q have been normalized.

That is, consider $P = \begin{bmatrix} 1 & 1 & * \\ 0 & 1 & * \\ -1 & 1 & * \end{bmatrix}$ then let $\mathbf{p}_3 = \mathbf{x}_3$. So $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.

Then $P \longrightarrow I$. That is, the columns of P are linearly independent.

Applying the Gram-Schmidt Process, we have:

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{q}_1 \cdot \mathbf{x}_3}{\mathbf{q}_1 \cdot \mathbf{q}_1}\right)\mathbf{q}_1 - \left(\frac{\mathbf{q}_2 \cdot \mathbf{x}_3}{\mathbf{q}_2 \cdot \mathbf{q}_2}\right)\mathbf{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{\sqrt{3}}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

In order for Q to be an orthogonal matrix, its columns must be orthonormal.
Therefore, we must compute the length of $\mathbf{x}_3$ and divide each of the components by it.

Clearly, $\|\mathbf{x}_3\| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 \left(-\frac{1}{2}\right)^2} = \frac{\sqrt{6}}{2}$.

Therefore, the completed matrix $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$.

Q: Why is Q an orthogonal matrix?
A: Because it has orthonormal columns.

Q: Therefore, what two conditions do the columns of Q satisfy?
A: For a set to be orthonormal it must have two key properties:

$$\mathbf{q}_i \cdot \mathbf{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{Property 1} \\ 1 & \text{if } i = j \quad \text{Property 2} \end{cases}$$

Verify this.

Q: Do the rows of Q also form an orthonormal set?
A: According to Theorem 5.7 of Section 5.1, they should. Verify this.
This verification also provides a quick check on our final result.

Q: If we have an orthogonal set can we divide by the lengths to create an orthonormal set?
A: Yes. Why?
If $\mathbf{v} \cdot c\mathbf{w} = \mathbf{0}$ $(c \neq 0)$, then $\mathbf{v} \cdot \mathbf{w} = \mathbf{0}$.
Why is this sufficient? Verify that this claim is true.

15. From 9, $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{v}_2\| = \frac{\sqrt{6}}{2}$, $\|\mathbf{v}_3\| = \frac{2\sqrt{3}}{3} \Rightarrow Q = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix}$.

Therefore, $R = Q^TA = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$.

Verify that $A = QR = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

16. From 10, $\|\mathbf{v}_1\| = 2$, $\|\mathbf{v}_2\| = 2\sqrt{6}$, $\|\mathbf{v}_3\| = \sqrt{2} \Rightarrow Q = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & -1/\sqrt{6} & 1/\sqrt{2} \\ -1/2 & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/2 & 2/\sqrt{6} & 0 \end{bmatrix}$.

Therefore, $R = Q^TA = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2\sqrt{6} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

17. $R = Q^TA = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 1/3 \\ 0 & 6 & 2/3 \\ 0 & 0 & 7/3 \end{bmatrix}$.

Verify that $A = QR = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 9 & 1/3 \\ 0 & 6 & 2/3 \\ 0 & 0 & 7/3 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix}$.

18. $R = Q^TA = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{6} & 2\sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix}$.

Verify that $A = QR = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 2\sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$.

19. Since A is orthogonal, simply let $A = Q \Rightarrow A = AI$, where I is obviously upper triangular.

20. A invertible $\Rightarrow A$ has linearly independent columns, so Theorem 5.16 $\Rightarrow A = QR$.
$A = QR$, R upper triangular with nonzero diagonal entries $\Rightarrow Q, R$ invertible $\Rightarrow A$ invertible.

Note: $Q^{-1} = Q^T$, $A = QR \Rightarrow Q^TA = R \Rightarrow R^{-1}Q^TA = R^{-1}R = I \Rightarrow A^{-1} = R^{-1}Q^T$.