5.2 Orthogonal Complements and Projections

1. Since $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for W, for all $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $W^\perp$ we have $\mathbf{v} \cdot \mathbf{w} = x + 2y = 0$.

   So, $W^\perp = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0 \right\}$ which implies $x = -2y$. Therefore, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis.

   Note the following general pattern:
   The lines that describe the necessary condition to be in W and $W^\perp$ are necessarily perpendicular.
   So, in this case, $2x - y = 0$ and $x + 2y = 0$ are necessarily perpendicular.

2. $\mathbf{v} \cdot \mathbf{w} = 4x - 3y = 0 \Rightarrow W^\perp = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 4x - 3y = 0 \right\} \Rightarrow 4x = 3y \Rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a basis.

   Note the following general pattern:
   The lines that describe the necessary condition to be in W and $W^\perp$ are necessarily perpendicular.
   So, in this case, $3x + 4y = 0$ and $4x - 3y = 0$ are necessarily perpendicular.

3. W consists of $\begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

   So, for all $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $W^\perp$ we have $\begin{matrix} \mathbf{v} \cdot \mathbf{w}_1 = x + z = 0 \\ \mathbf{v} \cdot \mathbf{w}_2 = y + z = 0 \end{matrix}$ which imply $\begin{matrix} x = y \\ z = -x \end{matrix}$.

   Therefore, $W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = t, z = -t \right\}$ which has basis $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

   Note the following general pattern:
   If W is a plane with normal $\mathbf{n}$, then $W^\perp$ is a line that has direction vector $\mathbf{n}$.
   Likewise, if W is a line that has direction vector $\mathbf{n}$, then $W^\perp$ is a plane with normal $\mathbf{n}$.

4. W consists of $\begin{bmatrix} x \\ 2x + 3z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

   $\begin{matrix} \mathbf{v} \cdot \mathbf{w}_1 = x + 2y = 0 \\ \mathbf{v} \cdot \mathbf{w}_2 = 3y + z = 0 \end{matrix} \Rightarrow W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -2t, y = t, z = -3t \right\}$ with basis $\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$.

   Note the following general pattern:
   If W is a plane with normal $\mathbf{n}$, then $W^\perp$ is a line that has direction vector $\mathbf{n}$.
   Likewise, if W is a line that has direction vector $\mathbf{n}$, then $W^\perp$ is a plane with normal $\mathbf{n}$.

5. Since $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ is a basis for W, for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $W^{\perp}$, $\mathbf{v} \cdot \mathbf{w} = x - y + 3z = 0$.

Therefore, $W^{\perp} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 3z = 0 \right\}$, which implies $y = x + 3z$.

So, $W^{\perp}$ consists of $\begin{bmatrix} x \\ x+3z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for $W^{\perp}$.

6. Since $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is a basis for W, for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $W^{\perp}$ $\mathbf{v} \cdot \mathbf{w} = 2x + 2y - z = 0 \Rightarrow$

$W^{\perp} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + 2y - z = 0 \right\} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $W^{\perp}$.

7. Since $A \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\{ [\, 1 \;\; 0 \;\; 1 \,], [\, 0 \;\; 1 \;\; -2 \,] \}$ is a basis for $\text{row}(A)$.

Since $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}$ in $\text{null}(A) = \begin{bmatrix} -x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$, $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{null}(A)$.

$[\, 1 \;\; 0 \;\; 1 \,] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0$, $[\, 0 \;\; 1 \;\; -2 \,] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0 \Rightarrow \mathbf{v}$ in $\text{row}(A) \perp \mathbf{x}$ in $\text{null}(A)$.

10. $A^T \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{row}(A^T) = \text{col}(A)$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{y} = \begin{bmatrix} -y_4 \\ -y_4 \\ y_4 \\ y_4 \end{bmatrix}, \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ is a basis for $\text{null}(A^T)$.

We now verify that the basis vectors for $\text{col}(A)$ and $\text{null}(A^T)$ are orthogonal:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = -1 + 1 = 0. \qquad \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = -1 + 1 = 0.$$

$$\begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = -1 + 1 = 0.$$

11. $[\,A \mid \mathbf{0}\,] = \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right] \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ -10x_1 \\ -4x_1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix} \right\}$ is a basis for $W^\perp$.

12. $[\,A \mid \mathbf{0}\,] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 + x_3 - x_4 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{array} \Rightarrow \begin{array}{c} x_1 = x_4 - x_3 \\ x_2 = -x_4 - 2x_3 \end{array} \Rightarrow$

Every $\mathbf{x}$ in $\text{null}(A)$ is of the form $\begin{bmatrix} x_4 - x_3 \\ -x_4 - 2x_3 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $W^\perp$.

13. $\left[\begin{array}{cccc|c} 2 & -1 & 6 & 3 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} 2x_1 - x_2 + 6x_3 + 3x_4 = 0 \\ 3x_2 - x_4 = 0 \end{array} \Rightarrow \begin{array}{c} x_1 = -4x_2 - 3x_3 \\ x_4 = 3x_2 \end{array} \Rightarrow$

Every $\mathbf{x}$ in $\text{null}(A) = \begin{bmatrix} -4x_2 - 3x_3 \\ x_2 \\ x_3 \\ 3x_2 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $W^\perp$.

14. $\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & 8 & 0 \\ 0 & 2 & 1 & 3 & -11 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 - x_3 - 2x_4 + 8x_5 = 0 \\ 2x_2 + x_3 + 3x_4 - 11x_5 = 0 \\ x_3 - x_5 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -7x_3 + 2x_4 \\ x_2 = 5x_3 - \frac{3}{2}x_4 \\ x_5 = x_3 \end{array} \Rightarrow$

Every $\mathbf{x}$ in null$(A) = \begin{bmatrix} -7x_3 + 2x_4 \\ 5x_3 - \frac{3}{2}x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -7 \\ 5 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -\frac{3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $W^\perp$.

15. So, $\mathbf{u}_1 \cdot \mathbf{v} = 3$, $\mathbf{u}_1 \cdot \mathbf{u}_1 = 2$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{proj}_W(\mathbf{v}) = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$.

16. $\mathbf{u}_1 \cdot \mathbf{v} = 2$, $\mathbf{u}_2 \cdot \mathbf{v} = 2$, $\mathbf{u}_1 \cdot \mathbf{u}_1 = 3$, $\mathbf{u}_2 \cdot \mathbf{u}_2 = 2 \Rightarrow \text{proj}_W(\mathbf{v}) = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ 2/3 \end{bmatrix}$.

17. $\text{proj}_W(\mathbf{v}) = \frac{1}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{13}{18}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix}$.

18. $\text{proj}_W(\mathbf{v}) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

19. $\text{proj}_W(\mathbf{v}) + (\mathbf{v} - \text{proj}_W(\mathbf{v})) = -\frac{4}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \left(\begin{bmatrix} 2 \\ -2 \end{bmatrix} - \frac{-4}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2/5 \\ -6/5 \end{bmatrix} + \begin{bmatrix} 12/5 \\ -4/5 \end{bmatrix}$.

20. $\text{proj}_W(\mathbf{v}) + (\mathbf{v} - \text{proj}_W(\mathbf{v})) = \frac{3}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 7/2 \\ -3 \\ 5/2 \end{bmatrix}$.

21. $\left(\frac{3}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{9}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) + \left(\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} - \left(\frac{3}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{9}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)\right) = \begin{bmatrix} 7/2 \\ -2 \\ 7/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$.

22. $\left(\frac{6}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{9}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) + \left(\begin{bmatrix} 2 \\ 1 \\ 5 \\ 3 \end{bmatrix} - \left(\frac{6}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{9}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)\right) = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

28. Since this is an if and only if statement, there are two statements to prove.
Let $\{\mathbf{u}_k\}$ be an orthogonal basis for W.

if: If $\text{proj}_W(\mathbf{x}) = \mathbf{0}$ then $\mathbf{x}$ is orthogonal to W.
That is, if $\text{proj}_W(\mathbf{x}) = \mathbf{0}$ then $\mathbf{x}$ is in $W^{\perp}$.
That is, $\mathbf{v} \cdot \mathbf{x} = 0$ for all $\mathbf{v}$ in W.
If $\sum_{k=1}^{n} (\mathbf{u}_k \cdot \mathbf{x})\mathbf{u}_k = \mathbf{0}$, then $\mathbf{u}_k \cdot \mathbf{x} = 0$ for all $\mathbf{u}_k$ in the basis for W.
Therefore, $\mathbf{v} \cdot \mathbf{x} = 0$ for all $\mathbf{v}$ in W as required. Why?
If $\mathbf{v}$ is in W then $\mathbf{v} = \sum_{k=1}^{n} \beta_k \mathbf{u}_k$, but $\mathbf{v} \cdot \mathbf{x} = \left(\sum_{k=1}^{n} \beta_k \mathbf{u}_k \right) \cdot \mathbf{x} = \sum_{k=1}^{n} \beta_k (\mathbf{u}_k \cdot \mathbf{x}) = 0$.

only if: If $\mathbf{x}$ is orthogonal to W, then $\text{proj}_W(\mathbf{x}) = \mathbf{0}$.
If $\mathbf{x}$ is orthogonal to W, then $\mathbf{v} \cdot \mathbf{x} = 0$ for all $\mathbf{v}$ in W.
So, if $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is the basis for W, then $\mathbf{u}_i \cdot \mathbf{x} = 0$ for all i.
So, $\text{proj}_W(\mathbf{x}) = \sum_{i=1}^{k} \left(\frac{\mathbf{u}_i \cdot \mathbf{x}}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i = \sum_{i=1}^{k} \left(\frac{0}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i = \mathbf{0}$ as required.

29. We will use our insights from Exercise 27 to prove this assertion.
Note that $\text{proj}_W(\mathbf{x})$ is in W. Why?

Because $\text{proj}_W(\mathbf{x}) = \sum_{k=1}^{n} (\mathbf{u}_k \cdot \mathbf{x})\mathbf{u}_k$, so $\text{proj}_W(\mathbf{x})$ is in $\text{span}(\mathbf{u}_k) = W$.

By Exercise 27, if $\mathbf{v}$ is in W then $\text{proj}_W(\mathbf{v}) = \mathbf{v}$.
Since $\mathbf{v} = \text{proj}_W(\mathbf{x})$ is in W, $\text{proj}_W(\text{proj}_W(\mathbf{x})) = \text{proj}_W(\mathbf{x})$.