5.1 Orthogonality in $\mathbb{R}^n$

1. $\mathbf{v}_1 \cdot \mathbf{v}_2 = (-3)(2) + 1(4) + 2(1) = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 2(1) + 4(-1) + 1(2) = 0$,
$\mathbf{v}_1 \cdot \mathbf{v}_3 = (-3) + 1(-1) + 2(2) = 0 \Rightarrow$ This set of vectors is orthogonal.

2. $\mathbf{v}_1 \cdot \mathbf{v}_2 = 4(-1) + 2(2) + (-5)(0) = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = (-1)(2) + 2(1) + 0(2) = 0$,
$\mathbf{v}_1 \cdot \mathbf{v}_3 = 4(2) + 2(1) + (-5)(2) = 0 \Rightarrow$ This set of vectors is orthogonal.

3. $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3(-1) + 1(2) + (-1)(1) \neq 0 \Rightarrow$ *not* orthogonal.

4. $\mathbf{v}_1 \cdot \mathbf{v}_3 = 5(3) + 3(1) + 1(-1) \neq 0 \Rightarrow$ *not* orthogonal.

5. $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 = 0 \Rightarrow$ This set of vectors is orthogonal.

6. $\mathbf{v}_2 \cdot \mathbf{v}_4 = 0(1) + (-1)(0) + 1(1) + 1(2) \neq 0 \Rightarrow$ *not* orthogonal.

7. $c_1 = \frac{4+6}{16+4} = \frac{1}{2}$, $c_2 = \frac{1-6}{1+4} = -1 \Rightarrow [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$. $\mathbf{w} = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

8. $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \Rightarrow c_1 = \frac{1+3}{1+9} = \frac{2}{5}$, $c_2 = \frac{-6+2}{36+4} = -\frac{1}{10} \Rightarrow [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2/5 \\ -1/10 \end{bmatrix}$.

9. $c_1 = \frac{1+0-1}{1+0+1} = 0$, $c_2 = \frac{1+2+1}{1+4+1} = \frac{2}{3}$, $c_3 = \frac{1-1+1}{1+1+1} = \frac{1}{3} \Rightarrow [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix}$.

10. $c_1 = \frac{1+2+3}{1+1+1} = 2$, $c_2 = \frac{1-2+0}{1+1+0} = -\frac{1}{2}$, $c_3 = \frac{1+2-6}{1+1+4} = -\frac{1}{2} \Rightarrow [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1/2 \\ -1/2 \end{bmatrix}$.

11. $\|\mathbf{v}_1\| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1$, $\|\mathbf{v}_2\| = \sqrt{(-\frac{4}{5})^2 + (\frac{3}{5})^2} = 1 \Rightarrow$ This set is orthonormal.

12. $\|\mathbf{v}_1\| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{1}{\sqrt{2}}$, $\|\mathbf{v}_2\| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2} = \frac{1}{\sqrt{2}}$
$\Rightarrow$ We need to normalize both vectors.
So, multiply each of them by $\frac{1}{length} = \sqrt{2} \Rightarrow \left\{ \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} \right\}$.

13. $\|\mathbf{v}_1\| = \sqrt{(\frac{1}{3})^2 + 2(\frac{2}{3})^2} = 1$, $\|\mathbf{v}_2\| = \sqrt{(\frac{2}{3})^2 + (-\frac{1}{3})^2} = \frac{\sqrt{5}}{3}$, $\|\mathbf{v}_3\| = \sqrt{(1^2 + 2^2 + (-\frac{5}{2})^2} = \frac{3\sqrt{5}}{2}$
$\Rightarrow \left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \frac{3}{\sqrt{5}} \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \frac{2}{3\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} 2/3\sqrt{5} \\ 4/3\sqrt{5} \\ -5/3\sqrt{5} \end{bmatrix} \right\}$.

14. $\|\mathbf{v}_1\| = 1, \|\mathbf{v}_2\| = \frac{2}{\sqrt{6}}$, $\|\mathbf{v}_3\| = \frac{1}{\sqrt{3}} \Rightarrow$
$\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \frac{\sqrt{6}}{2} \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}, \sqrt{3} \begin{bmatrix} 1/2 \\ -1/6 \\ 1/6 \\ -1/6 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ -\sqrt{3}/6 \end{bmatrix} \right\}$.

15. $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = \|\mathbf{v}_4\| = 1 \Rightarrow$ orthonormal.

16. $QQ^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ orthogonal, $Q^{-1} = Q^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

17. $QQ^T = I \Rightarrow Q$ is orthogonal and $Q^{-1} = Q^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

18. $\|\mathbf{q}_2\| = \sqrt{(\frac{1}{3})^2 + (-\frac{1}{3})^2 + 0^2} \neq 1 \Rightarrow Q$ is *not* orthogonal (columns must have length 1).

19. $QQ^T = I \Rightarrow Q$ is orthogonal and $Q^{-1} = Q^T = \begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$.

20. $QQ^T = I \Rightarrow Q$ is orthogonal and $Q^{-1} = Q^T = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$.

21. $\mathbf{q}_1 \cdot \mathbf{q}_4 = 1(\frac{1}{\sqrt{6}}) + 0(\frac{1}{\sqrt{6}}) + 0(-\frac{1}{\sqrt{6}}) + 0(-\frac{1}{\sqrt{2}}) \neq 0 \Rightarrow$ not orthogonal (columns must be $\perp$).

22. By Theorem 5.6(c), we need only show that $Q^{-1}\mathbf{x} \cdot Q^{-1}\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$, for every $\mathbf{x}$ and $\mathbf{y}$.
We will show this using Theorem 5.5: Q is orthogonal $\Leftrightarrow Q^{-1} = Q^T \Leftrightarrow QQ^T = I$.
$Q^{-1}\mathbf{x} \cdot Q^{-1}\mathbf{y} = Q^T\mathbf{x} \cdot Q^T\mathbf{y} = (Q^T\mathbf{x})^T Q^T\mathbf{y} = \mathbf{x}^T QQ^T\mathbf{y} = \mathbf{x}^T I\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

23. We will show this using the fact that $\det I = 1, QQ^T = I$, and $\det Q = \det Q^T$.
$1 = \det I = \det(QQ^T) = \det Q \det Q^T = \det Q \det Q \Rightarrow \sqrt{(\det Q)^2} = \sqrt{1} \Rightarrow \det Q = \pm 1$.

24. By Theorem 5.5, we need only show $(Q_1Q_2)^{-1} = (Q_1Q_2)^T$, i.e. $(Q_1Q_2)^T Q_1Q_2 = I$.
$(Q_1Q_2)^T Q_1Q_2 = Q_2^T(Q_1^TQ_1)Q_2 = Q_2^T I Q_2 = Q_2^TQ_2 = I$.

25. Induction on Exercise 24 $\Rightarrow Q_1, Q_2, \ldots, Q_n$ orthogonal $\Rightarrow Q = Q_n \ldots Q_2Q_1$ orthogonal.
Let $P = P_n \ldots P_2P_1$, where P_k is an elementary matrix corresponding to a row interchange.
Then $P_kP_k^T = I$ and Theorem 5.5 $\Rightarrow P_k$ is orthogonal $\Rightarrow P = P_n \ldots P_2P_1$ is orthogonal.

26. Let $Q' = PQ$, be obtained by rearranging the rows of Q, so P is a permutation matrix.
Then by Exercise 25, the permutation matrix P is orthogonal. Therefore, Exercise 24 in which we proved Theorem 5.8(d) $\Rightarrow Q' = PQ$ is orthogonal because P and Q are orthogonal.

27. Let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$ and ϕ be the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$.
We need to show that $\theta = \phi$. We will use Theorem 6(b,c): $\|Q\mathbf{x}\| = \|\mathbf{x}\|, Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.
$0 \leq \theta, \phi \leq \pi \Rightarrow \cos\theta = \cos\phi \Rightarrow \theta = \phi$. So $\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{Q\mathbf{x}\cdot Q\mathbf{y}}{\|Q\mathbf{x}\|\|Q\mathbf{y}\|} = \cos\phi \Rightarrow \theta = \phi$.

28. (a) $QQ^T = I \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow a^2+b^2=1 \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector.

Now all we have left to show is ($d=a$ and $c=-b$) or ($d=-a$ and $c=b$).
From the fact that $QQ^T = I$, we have these three equations: $a^2+b^2=1 \Rightarrow b^2 = 1-a^2$ (1), $c^2+d^2=1 \Rightarrow c^2 = 1-d^2$ (2), $ac+bd=0 \Rightarrow a^2c^2 = b^2d^2$ (3).
Substituting (1), (2) into (3) $\Rightarrow a^2(1-d^2) = (1-a^2)d^2 \Rightarrow a^2 = d^2 \Rightarrow a = \pm d$.
When $d=a=0 \Rightarrow b=c=\pm 1$, we have the following four possibilities:
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
all of which have one of the two forms listed.
When $d = a \neq 0$, we have:

$$\begin{bmatrix} a & c \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow ab+ac=0 \Rightarrow ac=-ab \Rightarrow c=-b.$$

So, we get $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Likewise, when $d = -a \neq 0$, we have:

$$\begin{bmatrix} a & c \\ b & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow ab-ac=0 \Rightarrow ac=ab \Rightarrow c=b.$$

So, we get $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

(b) $\begin{bmatrix} a \\ b \end{bmatrix}$ unit vector $\Rightarrow 0 \le |a|, |b| \le 1, a^2+b^2=1 \Rightarrow$ There exists θ with $a=\cos\theta, b=\sin\theta$.

(c) Let $\mathbf{x} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$, $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, and $R = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Then we have:

$$Q\mathbf{x} = \begin{bmatrix} r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ r(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{bmatrix} = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix} \Rightarrow Q \text{ represents a rotation,}$$

$$R\mathbf{x} = \begin{bmatrix} r(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ r(\sin\theta\cos\phi - \cos\theta\sin\phi) \end{bmatrix} = \begin{bmatrix} r\cos(\phi-\theta) \\ -r\sin(\phi-\theta) \end{bmatrix} \Rightarrow R \text{ represents a reflection.}$$

(d) We will show this using the fact that $\cos^2\theta + \sin^2\theta = 1$. So, we have:
$\det Q = \cos^2\theta + \sin^2\theta = 1$ and $\det R = -\cos^2\theta - \sin^2\theta = -(\cos^2\theta + \sin^2\theta) = -1$.

29. By Exercise 28(d), $\det Q = 1 \Rightarrow$ rotation. So, since $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$, $\theta = \frac{\pi}{4}$ or $45°$.

30. $\det Q = 1 \Rightarrow$ rotation. Furthermore, since $\cos\theta = -\frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$, $\theta = \frac{4\pi}{3}$ or $240°$.

31. $\det Q = -1 \Rightarrow$ reflection. Line of reflection is $y = -\tan\theta\, x = -\frac{\sin\theta}{\cos\theta}\, x = -(\frac{\sqrt{3}/2}{-1/2})\, x = \sqrt{3}\, x$.

32. $\det Q = -1 \Rightarrow$ reflection. Line of reflection is $y = -\tan\theta\, x = -\frac{\sin\theta}{\cos\theta}\, x = -(\frac{-4/5}{-3/5})\, x = -\frac{4}{3}\, x$.

33. (a) $\Rightarrow AA^T = I$, $B^TB = I \Rightarrow A(A^T + B^T)B = AA^TB + AB^TB = IB + AI = A + B$.

(b) Note: B orthogonal and Theorem 5.8(b) $\Rightarrow \det B = \pm 1 \Rightarrow \det B \det B = 1$.
If $\det A + \det B = 0$ also, then $\det A = -\det B \Rightarrow \det A \det B = -\det B \det B = -1$.
Also, recall: $\det(AB) = \det A \det B$ and $\det(A^T + B^T) = \det(A + B)^T = \det(A + B)$.
In order to prove $A + B$ is not invertible, we need only show that $\det(A + B) = 0$.
$A+B = A(A^T+B^T)B \Rightarrow \det(A+B) = \det(A(A^T+B^T)B) = \det A \det B \det(A^T+B^T)$.
Now use the fact that A, B orthogonal and $\det A + \det B = 0 \Rightarrow \det A \det B = -1$.
So, $\det(A + B) = \det A \det B \det(A^T + B^T) = -\det(A^T + B^T) = -\det(A + B)$.
Therefore, $\det(A + B) = -\det(A + B) \Rightarrow 2\det(A + B) = 0 \Rightarrow \det(A + B) = 0 \Rightarrow$
$A + B$ is not invertible.

34. We will show $QQ^T = I \Rightarrow Q$ is orthogonal.

The fact that $\mathbf{x}$ is a unit vector $\Rightarrow x_1^2 + \ldots + x_n^2 = 1 \Rightarrow x_2^2 + \ldots + x_n^2 = 1 - x_1^2$.
That is, $\mathbf{y}^T\mathbf{y} = x_2^2 + \ldots + x_n^2 = 1 - x_1^2 \Rightarrow x_1^2 + \mathbf{y}^T\mathbf{y} = 1$.

So, $\left(\frac{1}{1-x_1}\right)\mathbf{y}^T\mathbf{y} = \frac{x_2^2+\ldots+x_n^2}{1-x_1} = \frac{1-x_1^2}{1-x_1} = 1 + x_1$. Also:

$$\mathbf{y}^T\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right) = \mathbf{y}^T - \left(\frac{1}{1-x_1}\right)(\mathbf{y}^T\mathbf{y})\mathbf{y}^T = \mathbf{y}^T - \frac{1-x_1^2}{1-x_1}\mathbf{y}^T = \mathbf{y}^T - (1+x_1)\mathbf{y}^T = -x_1\mathbf{y}^T.$$

Similarly, $\left(I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right)\mathbf{y} = \mathbf{y} - \left(\frac{1}{1-x_1}\right)(\mathbf{y}(\mathbf{y}^T\mathbf{y}) = \mathbf{y} - \frac{1-x_1^2}{1-x_1}\mathbf{y} = \mathbf{y} - (1+x_1)\mathbf{y} = -x_1\mathbf{y}$.

Finally, $\left[I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right]\left[I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T\right] = I^2 - 2\left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T + \left(\frac{1}{1-x_1}\right)^2\mathbf{y}(\mathbf{y}^T\mathbf{y})\mathbf{y}^T$
$= I - \frac{2-2x_1}{(1-x_1)^2}\mathbf{y}\mathbf{y}^T + \frac{1-x_1^2}{(1-x_1)^2}\mathbf{y}\mathbf{y}^T = I - \frac{1-2x_1+x_1^2}{(1-x_1)^2}\mathbf{y}\mathbf{y}^T = I - \mathbf{y}\mathbf{y}^T$.

$$\text{So, } QQ^T = \left[\begin{array}{c|c} x_1 & \mathbf{y}^T \\ \hline \mathbf{y} & I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T \end{array}\right]\left[\begin{array}{c|c} x_1 & \mathbf{y}^T \\ \hline \mathbf{y} & I - \left(\frac{1}{1-x_1}\right)\mathbf{y}\mathbf{y}^T \end{array}\right]$$

$$= \left[\begin{array}{c|c} x_1^2 + \mathbf{y}^T\mathbf{y} = 1 & x_1\mathbf{y}^T - x_1\mathbf{y}^T = 0 \\ \hline x_1\mathbf{y} - x_1\mathbf{y} = 0 & \mathbf{y}\mathbf{y}^T + (I - \mathbf{y}\mathbf{y}^T) = I \end{array}\right] = I.$$