# 4.4 Similarity and Diagonalization

**1.** $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 1$, while

$\det(B-\lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1$. Thus, by Theorem 4.22(d), $A \not\sim B$.

**2.** $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ -5 & 7-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 16$, while

$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16$. Thus, by Theorem 4.22(d), $A \not\sim B$.

**3.** $A$ has eigenvalues 2, 2, 4; $B$ has eigenvalues 1, 4, and 4 $\Rightarrow A \not\sim B$.

**4.** $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 3\lambda$, while

$\det(B-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - 3\lambda \Rightarrow A \not\sim B$.

**5.** Since $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ is of the form $P^{-1}AP = D$,

we see that the eigenvalues of $A$ are $\lambda_1 = 4$ and $\lambda_2 = 3$.

So, $E_4$, $E_3$ have bases $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

**6.** The eigenvalues of $A$ are 2, 0, $-1 \Rightarrow E_2$, $E_0$, $E_{-1}$ have bases $\mathbf{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

**7.** The eigenvalues of $A$ are 6 and $-2 \Rightarrow E_6$, $E_{-2}$ have bases $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

8. $\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 21 = (\lambda-3)(\lambda-7)$,
so $A$ has eigenvalues 3 and 7 and is thus diagonalizable, by Theorem 4.25.
We find bases for the eigenspaces:
$A - 3I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and
$A - 7I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, so $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Thus $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ diagonalizes $A$, and
$P^{-1}AP = D \Leftrightarrow AP = PD \Leftrightarrow \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & 7 \end{bmatrix}$.

9. $\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$,
so $A$ has eigenvalue $-1$ with algebraic multiplicity 2.
To find the corresponding eigenspace, we calculate
$A + I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$, so the eigenvalue $-1$ has geometric multiplicity 1,
and thus $A$ is not diagonalizable, by the Diagonalization Theorem.

10. $A$ has eigenvalue 3 with algebraic multiplicity 3.
Furthermore $A - 3I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so this eigenvalue has geometric multiplicity 1,
and thus $A$ is not diagonalizable, by the Diagonalization Theorem.

**14.** $A$ has eigenvalues 1, 2, and 3. Since the eigenvalue 3 has algebraic multiplicity 2, we check $A - 3I = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus 3 has geometric multiplicity 1, and $A$ is thus not diagonalizable.

**15.** $A$ has eigenvalues $-2$ and 2. We check

$$A + 2I = \begin{bmatrix} 4 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{-2} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } E_{2} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus $P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ satisfy $P^{-1}AP = D$.

**16.** We diagonalize $A$, first finding its eigenvalues and eigenvectors:

$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$, so $A$ has eigenvalues $-1$, 2.

$A + I = \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$, so $E_{-1}$ has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$A - 2I = \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, so $E_{2}$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Therefore, $P^{-1}AP = D$ is satisfied if $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. So

$$\begin{aligned} \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}^9 &= \left(PDP^{-1}\right)^9 = PD^9P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^9 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^9 & 0 \\ 0 & 2^9 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -514 & 1026 \\ -513 & 1025 \end{bmatrix}. \end{aligned}$$

17. We diagonalize $A$, first finding its eigenvalues and eigenvectors:

$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 6 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$, so $A$ has eigenvalues $-3$ and 2.

$A+3I = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, so $E_{-3}$ has basis $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

$A-2I = \begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$, so $E_2$ has basis $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Therefore, $P^{-1}AP = D$ is satisfied if $P = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$. Therefore,

$$\begin{aligned} \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}^{10} &= (PDP^{-1})^{10} = PD^{10}P^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}^{10}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-3)^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 35,839 & -69,630 \\ -11,605 & 24,234 \end{bmatrix}. \end{aligned}$$

18. $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2-6\lambda+5 = (\lambda-1)(\lambda-5)$.

$A-I = \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \Rightarrow E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $A-5I = \begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \Rightarrow E_5 = \text{span}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$.

Therefore

$$\begin{aligned} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}^{-6} &= (PDP^{-1})^{-6} = PD^{-6}P^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1^{-6} & 0 \\ 0 & 5^{-6} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{15,625} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} = \frac{1}{15,625}\begin{bmatrix} 3907 & 11,718 \\ 3906 & 11,719 \end{bmatrix}. \end{aligned}$$

19. $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2-2\lambda-3 = (\lambda+1)(\lambda-3)$.

$A+I = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \Rightarrow E_{-1} = \text{span}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$ and $A-3I = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \Rightarrow E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

Therefore

$$\begin{aligned} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}^{k} &= (PDP^{-1})^{k} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}^{k}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & 3^k \end{bmatrix}\frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3(-1)^k+3^k & 3(-1)^{k+1}+3^{k+1} \\ (-1)^{k+1}+3^k & (-1)^k+3^{k+1} \end{bmatrix}. \end{aligned}$$

**20.** $\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 1 & 2\\ 2 & 1-\lambda & 2\\ 2 & 1 & 2-\lambda\end{vmatrix}=-\lambda^2(\lambda-5)$. $E_0=\text{span}\left(\begin{bmatrix}1\\-2\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$ and

$A-5I=\begin{bmatrix}-3 & 1 & 2\\ 2 & -4 & 2\\ 2 & 1 & -3\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$, so $E_5=\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$. Therefore

$$\begin{bmatrix}2 & 1 & 2\\ 2 & 1 & 2\\ 2 & 1 & 2\end{bmatrix}^8=\begin{bmatrix}1 & 1 & 1\\ -2 & 0 & 1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 5\end{bmatrix}^8\begin{bmatrix}1 & 1 & 1\\ -2 & 0 & 1\\ 0 & -1 & 1\end{bmatrix}^{-1}$$
$$=\begin{bmatrix}0 & 0 & 5^8\\ 0 & 0 & 5^8\\ 0 & 0 & 5^8\end{bmatrix}\frac{1}{5}\begin{bmatrix}1 & -2 & 1\\ 2 & 1 & -3\\ 2 & 1 & 2\end{bmatrix}=5^7\begin{bmatrix}2 & 1 & 2\\ 2 & 1 & 2\\ 2 & 1 & 2\end{bmatrix}.$$

**21.** Since $D=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$, we have

$$A^{2002}=\left(PDP^{-1}\right)^{2002}=P\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}^{2002}P^{-1}=PP^{-1}=I_3.$$

**22.** $\det(A-\lambda I)=\begin{vmatrix}2-\lambda & 0 & 1\\ 1 & 1-\lambda & 1\\ 1 & 0 & 2-\lambda\end{vmatrix}=-\lambda^3+5\lambda^2-7\lambda+3=-(\lambda-1)^2(\lambda-3)$.

$E_1=\text{span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ and $A-3I=\begin{bmatrix}-1 & 0 & 1\\ 1 & -2 & 1\\ 1 & 0 & -1\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$,

so $E_3=\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$. Therefore

$$\begin{bmatrix}2 & 0 & 1\\ 1 & 1 & 1\\ 1 & 0 & 2\end{bmatrix}^{-5}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}^{-5}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{bmatrix}^{-1}$$
$$=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{243}\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 0 & -1\\ -1 & 2 & -1\\ 1 & 0 & 1\end{bmatrix}=\frac{1}{243}\begin{bmatrix}122 & 0 & -121\\ -121 & 243 & -121\\ -121 & 0 & 122\end{bmatrix}.$$

**23.** $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & -2-\lambda & 2 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3 + 7\lambda - 6 = -(\lambda+3)(\lambda-1)(\lambda-2).$

$$A+3I = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{-3} = \text{span}\left(\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}\right).$$

$$A-I = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right).$$

$$A-2I = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right). \text{ Therefore}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 & 1 \\ -4 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^k \begin{bmatrix} 1 & 1 & 1 \\ -4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{10} \begin{bmatrix} 1 & 1 & 1 \\ -4 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 5 & 0 & -5 \\ 4 & 2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 4(2)^k + (-3)^k + 5 & 2(2)^k - 2(-3)^k & 4(2)^k + (-3)^k - 5 \\ 4(2)^k - 4(-3)^k & 2(2)^k + 8(-3)^k & 4(2)^k - 4(-3)^k \\ 4(2)^k + (-3)^k - 5 & 2(2)^k - 2(-3)^k & 4(2)^k + (-3)^k + 5 \end{bmatrix}.$$

**24.** $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & k-\lambda \end{vmatrix} = k - \lambda - k\lambda + \lambda^2 = (\lambda-1)(\lambda-k).$

So $A$ is certainly diagonalizable if $k \neq 1$.

$k=1$ then $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 1$ has geometric multiplicity $1 \Rightarrow A$ is not diagonalizable.

Thus, $A$ is diagonalizable provided $k \neq 1$.

**25.** $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & k \\ 0 & 1-\lambda \end{vmatrix}$. The only eigenvalue is 1, and $A - I = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$.

If $k=0$, then $A$ is already diagonal;
otherwise 1 has geometric multiplicity 1 and thus $A$ is not diagonalizable.
Thus, $A$ is diagonalizable only if $k=0$.

**26.** $\det(A-\lambda I) = \begin{vmatrix} k-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - k\lambda - 1 = 0 \Leftrightarrow \lambda = \frac{k \pm \sqrt{k^2+4}}{2} \Rightarrow$

There are two distinct eigenvalues for all $k \Rightarrow A$ is diagonalizable for all $k$.