3.6 Introduction to Linear Transformations

1. Since T is the linear transformation corresponding to matrix A, $T(\mathbf{x}) = A\mathbf{x}$. So:

$$T(\mathbf{u}) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

2. Since T is the linear transformation corresponding to matrix A, $T(\mathbf{x}) = A\mathbf{x}$. So:

$$T(\mathbf{u}) = \begin{bmatrix} 4 & 0 & -1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} 4 & 0 & -1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

3. We prove T is a linear transformation by showing that $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$.

Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then:

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} c_1x_1 + c_2x_2 + c_1y_1 + c_2y_2 \\ c_1x_1 + c_2x_2 - c_1y_1 - c_2y_2 \end{bmatrix} = \begin{bmatrix} c_1x_1 + c_1y_1 \\ c_1x_1 - c_1y_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 + c_2y_2 \\ c_2x_2 - c_2y_2 \end{bmatrix} \\ &= c_1\begin{bmatrix} x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 + y_2 \\ x_2 - y_2 \end{bmatrix} = c_1T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2). \end{aligned}$$

Therefore, we conclude that T is a linear transformation.

4. We prove T is a linear transformation by showing that $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$.

Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x+2y \\ 3x-4y \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} -c_1y_1 - c_2y_2 \\ c_1x_1 + c_2x_2 + 2c_1y_1 + 2c_2y_2 \\ 3c_1x_1 + 3c_2x_2 - 4c_1y_1 - 4c_2y_2 \end{bmatrix} = \begin{bmatrix} -c_1y_1 \\ c_1x_1 + 2c_1y_1 \\ 3c_1x_1 - 4c_1y_1 \end{bmatrix} + \begin{bmatrix} -c_2y_2 \\ c_2x_2 + 2c_2y_2 \\ 3c_2x_2 - 4c_2y_2 \end{bmatrix} \\ &= c_1\begin{bmatrix} -y_1 \\ x_1 + 2y_1 \\ 3x_1 - 4y_1 \end{bmatrix} + c_2\begin{bmatrix} -y_2 \\ x_2 + 2y_2 \\ 3x_2 - 4y_2 \end{bmatrix} = c_1T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2). \end{aligned}$$

So, this is indeed a linear transformation.

5. We prove T is a linear transformation by showing that $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$.

Let $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y+z \\ 2x+y-3z \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. Then

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1z_1 + c_2z_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} c_1x_1 + c_2x_2 - c_1y_1 - c_2y_2 + c_1z_1 + c_2z_2 \\ 2c_1x_1 + 2c_2x_2 + c_1y_1 + c_2y_2 - 3c_1z_1 - 3c_2z_2 \end{bmatrix} \\ &= \begin{bmatrix} c_1x_1 - c_1y_1 + c_1z_1 \\ 2c_1x_1 + c_1y_1 - 3c_1z_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 - c_2y_2 + c_2z_2 \\ 2c_2x_2 + c_2y_2 - 3c_2z_2 \end{bmatrix} \\ &= c_1\begin{bmatrix} x_1 - y_1 + z_1 \\ 2x_1 + y_1 - 3z_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 - y_2 + z_2 \\ 2x_2 + y_2 - 3z_2 \end{bmatrix} = c_1T\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c_2T\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2). \end{aligned}$$

6. We prove T is a linear transformation by showing that $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$.

Let $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ y+z \\ x+y \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. Then

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T\left(c_1\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ c_1z_1 + c_2z_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} c_1x_1 + c_2x_2 + c_1z_1 + c_2z_2 \\ c_1y_1 + c_2y_2 + c_1z_1 + c_2z_2 \\ c_1x_1 + c_2x_2 + c_1y_1 + c_2y_2 \end{bmatrix} = \begin{bmatrix} c_1x_1 + c_1z_1 \\ c_1y_1 + c_1z_1 \\ c_1x_1 + c_1y_1 \end{bmatrix} + \begin{bmatrix} c_2x_2 + c_2z_2 \\ c_2y_2 + c_2z_2 \\ c_2x_2 + c_2y_2 \end{bmatrix} \\ &= c_1\begin{bmatrix} x_1 + z_1 \\ y_1 + z_1 \\ x_1 + y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 + z_2 \\ y_2 + z_2 \\ x_2 + y_2 \end{bmatrix} = c_1T\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c_2T\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2). \end{aligned}$$

7. We prove T is *not* a linear transformation by showing that $T(c\mathbf{v}) \neq cT(\mathbf{v})$ (property (2) fails).

Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x^2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$T(c\mathbf{v}) = T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\begin{bmatrix} cx \\ cy \end{bmatrix} = \begin{bmatrix} cy \\ c^2x^2 \end{bmatrix} \neq \begin{bmatrix} cy \\ cx^2 \end{bmatrix} = c\begin{bmatrix} y \\ cx^2 \end{bmatrix} = cT\begin{bmatrix} x \\ y \end{bmatrix} = cT(\mathbf{v})$$

Since $T(c\mathbf{v}) \neq cT(\mathbf{v})$ (property (2) fails), T is *not* a linear transformation.

Q: Is $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ linear?

A: No, by a very similar argument to the one given above.
We should suspect both T and S are *not* linear because x^2 is *not* linear.

8. We prove T is *not* a linear transformation by showing that $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$.

Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| \\ |y| \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

while

$$T(\mathbf{v}_1) + T(\mathbf{v}_2) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} + T\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} |1| \\ 0 \end{bmatrix} + \begin{bmatrix} |-1| \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Since $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$, T is *not* a linear transformation.

Q: Should we have anticipated that T would not be linear?

A: Probably. Since the graph of $|x|$ is not a line.

9. We prove T is *not* a linear transformation by showing that $T(c\mathbf{v}) \neq cT(\mathbf{v})$ (property (2) fails).

Let $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$T(c\mathbf{v}) = T\begin{bmatrix} cx \\ cy \end{bmatrix} = \begin{bmatrix} cxcy \\ cx+cy \end{bmatrix} = c\begin{bmatrix} cxy \\ x+y \end{bmatrix} \neq c\begin{bmatrix} xy \\ x+y \end{bmatrix} = cT\begin{bmatrix} x \\ y \end{bmatrix} = cT(\mathbf{v})$$

Since $T(c\mathbf{v}) \neq cT(\mathbf{v})$ (property (2) fails), T is *not* a linear transformation.

Q: Is there any reason to suspect that T is not linear before completing the proof?

A: Yes, by a very similar argument to the one given above.
We should suspect T is *not* linear because xy is *not* linear.

14. As on p.212, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\mathbf{v}$.

We have $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ y+z \\ x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

So $T = T_A$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

15. As in Example 3.56, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\mathbf{v}$.

We have $F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

So we identify $F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ as the matrix performing the desired transformation.

16. As in Example 3.57, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\mathbf{v}$.

We have $R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(x-y) \\ \frac{1}{\sqrt{2}}(x+y) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} x + \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} y = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

So, $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is the matrix performing the desired transformation.

17. As in Example 3.57, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\mathbf{v}$.

We have $D\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 3 \end{bmatrix} y = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

So, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is the matrix performing the desired transformation.

18. As in Example 3.59, we confirm T is a linear transformation by finding A such that $T(\mathbf{v}) = A\mathbf{v}$.

Since $x = y \Rightarrow x - y = 0$, the direction vector for the line $y = x$ is $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So, from the formula given for A at the end of Example 3.59, we have:

$A = \frac{1}{d_1^2+d_2^2}\begin{bmatrix} d_1^2 & d_1d_2 \\ d_1d_2 & d_2^2 \end{bmatrix}$. So, in this case: $A = \frac{1}{1+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

19. Let $A_1 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and $C_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.

$A_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$: A_1 stretches vectors horizontally by a factor of k.

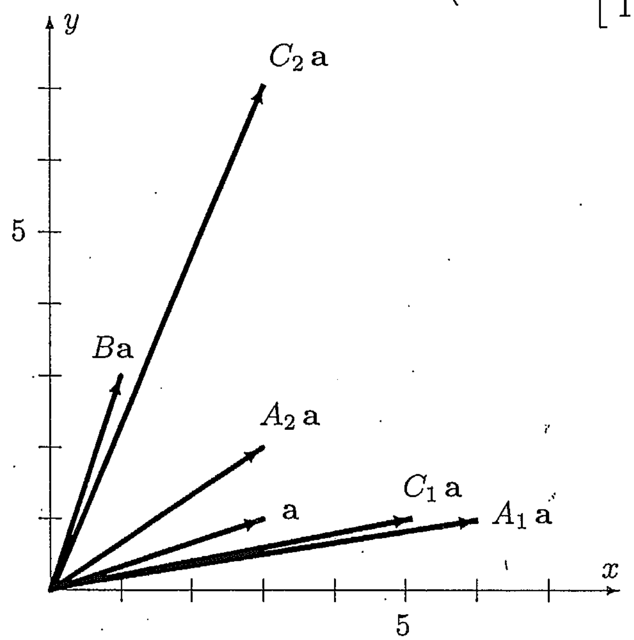
$A_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$: A_2 stretches vectors vertically by a factor of k.

$B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$: B reflects vectors in the line $y = x$.

$C_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$: C_1 extends vectors horizontally by ky.

$C_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y + kx \end{bmatrix}$: C_1 extends vectors vertically by kx.

The effects of these transformations are illustrated below (with $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $k = 2$).



28. We want to reflect about the line $y = \sqrt{3}x$ which lies in the direction of vector $\mathbf{d} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

so that $F_\ell(\mathbf{x}) = \frac{1}{1+3}\begin{bmatrix} 1-3 & 2(\sqrt{3}) \\ 2(\sqrt{3}) & -1+3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

Then $F_\ell(\mathbf{e}_1) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{bmatrix}$ and $F_\ell(\mathbf{e}_2) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} \\ \frac{1}{2} \end{bmatrix}$

and the standard matrix of this transformation is $A = \begin{bmatrix} F_\ell(\mathbf{e}_1) & F_\ell(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

29. We have the two linear transformations defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$ and

$S\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 + y_3 \\ 3y_2 - y_3 \\ y_1 - y_2 \\ y_1 + y_2 + y_3 \end{bmatrix}$. To find $(S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ directly, we calculate

$$\begin{aligned}(S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= S\begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 2(x_1) + (3x_1 + 4x_2) \\ 3(2x_1 - x_2) - (3x_1 + 4x_2) \\ x_1 - (2x_1 - x_2) \\ x_1 + (2x_1 - x_2) + (3x_1 + 4x_2) \end{bmatrix} \\ &= \begin{bmatrix} 5x_1 + 4x_2 \\ 3x_1 - 7x_2 \\ -x_1 + x_2 \\ 6x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & -7 \\ -1 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.\end{aligned}$$

Finally, we identify the standard matrix of $(S \circ T)$ as $A = \begin{bmatrix} 5 & 4 \\ 3 & -7 \\ -1 & 1 \\ 6 & 3 \end{bmatrix}$.

30. (a) $(S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = S\left(T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = S\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

So, by direct substitution we identify the matrix of $S \circ T$ as $\begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}$.

(b) We see that the standard matrices are $[S] = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and $[T] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

So, Theorem 3.32 gives $[S \circ T] = [S][T] = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}$

which is the same result as obtained by direct substitution.