Mathematics 214: Linear Systems
Fall 2006
Test 3: §§3.6, 4.1-4.4, 5.1

Name: ____________________________________

By signing below, you are agreeing on your honor that you are fully complying with the guidelines for this test and are not in any way violating the College's spirit of honor. In particular, you have not received or given any assistance on this test, nor will you give any assistance to those taking the test later.

Signature: ____________________________________

Please show your work on the problems neatly and completely. You must show the work you did and not just your answers. Please clearly identify your final answers.

No.	Score	Maximum
1		20
2		20
3		10
4		20
5		10
6		20
Total		**100**

1. [20 points] Find the following determinants. The following information may (or may not) be helpful:

$$\texttt{det}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5 \qquad \texttt{det}(A) = 2 \qquad \texttt{det}(BA^{-1}) = -6 \qquad \texttt{det}(C) = 3$$

(a) $\texttt{det}\begin{bmatrix} 0 & 2 & 0 & 0 \\ a & a & b & c \\ d & f & e & f \\ g & h & h & i \end{bmatrix} =$

(b) $\texttt{det}\begin{bmatrix} 2a-2d & 2b-2e & 2c-2f \\ -2d & -2e & -2f \\ 2g & 2h & 2i \end{bmatrix} =$

(c) $\texttt{det}\begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix} =$

(d) $\texttt{det}(B^T C^2) =$

2. [20 points] $\lambda_1 = 0$ and $\lambda_2 = -2$ are the only eigenvalues of $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. For each eigenvalue, find a basis for the corresponding eigenspace.

Is A invertible? Why or why not?

Is A diagonalizable? Why or why not?

3. [10 points] Put an "X" mark in the blank prior to all the statements below that are true for $n \times n$ orthogonal matrices Q. Do not mark those statements that are not true for all $n \times n$ orthogonal matrices.

______ (a) $Q^T Q = I_n$

______ (b) $Q^T = Q^{-1}$

______ (c) $\texttt{dim}(\texttt{null}(Q)) = 0$

______ (d) Q has n distinct eigenvalues

______ (e) $\lambda = 1$ is an eigenvalue of Q

______ (f) $\texttt{det}(Q) = \pm 1$

______ (g) Q^{-1} is an orthogonal matrix

______ (h) $\|Q\vec{e}_1\| = 1$ where $\vec{e}_1$ is the usual standard unit vector in $\mathbf{R}^n$

______ (i) The columns of Q span $\mathbf{R}^n$

______ (j) $Q\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbf{R}^n$

HAPPY THANKSGIVING! Oh, wait, there's more test to go ...

4. [20 points] Mark each of the following "TRUE" or "FALSE". If I can't tell whether you wrote "T" or "F", you will not get credit for your answer.

____________ (a) If A is invertible, then $\texttt{det}(A^{-1}) = \texttt{det}(A^T)$.

____________ (b) If the $n \times n$ matrix A is diagonalizable, then it has n distinct (different) eigenvalues.

____________ (c) The geometric multiplicity of an eigenvalue is always greater than or equal to its algebraic multiplicity.

____________ (d) The matrix transformation $A\vec{x} = \begin{bmatrix} 2 & -2 & 0 & 1 \\ 1 & 5 & -2 & 0 \end{bmatrix} \vec{x}$ is a linear transformation.

____________ (e) All the eigenvectors of A are in the nullspace of A.

____________ (f) $\texttt{det}(AB) = \texttt{det}(BA)$

____________ (g) If $\texttt{det}(A) \neq 0$, then A is invertible.

____________ (h) If $\lambda = 0$ is an eigenvalue of A, then A is not invertible.

____________ (i) If $\lambda = 0$ is an eigenvalue of A, then A is not diagonalizable.

____________ (j) A and B are similar if and only if B is a diagonal matrix, P is made up of the eigenvectors of A in its columns, and $P^{-1}AP = B$.

I know there's more still . . . But at least there's no homework for Math 214 over break!

5. [10 points] Prove **ONE** of the following statements. Circle the one you are proving.

(a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x+2y \\ 3x-4y \end{bmatrix}$ is / is not a linear transformation. (Note: You must first decide if it is or is not a linear transformation, and then prove that.) [§3.6 #4]

(b) If A is an invertible matrix with eigenvalue λ and corresponding eigenvector $\vec{x}$, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$. [§4.3 # 13]

(c) If A is diagonalizable, then so is A^T. [§4.4 # 41]

(d) If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$. [§5.1 # 24]

6. [20 points] The following is one possible factorization of the matrix A:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

(a) Show that the matrix $\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is orthogonal.

(b) Why is it obvious that $\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$?

(d) Write down the eigenvalue(s) for A? Next to each eigenvalue, write a basis for the corresponding eigenspace.

(e) Explain how this factorization might help you find A^{10} more quickly than actually multiplying A together 10 times. [You don't actually have to do the calculation, but an appropriate equation and a complete explanation is needed for full credit.]

(BONUS) Why might we call matrix A "orthogonally diagonalizable"?