MATH 214 FALL 2007

1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\quad rref(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

(a) $rank(A) = 2 = \dim(col(A)) = \dim(row(A))$

$\dim(null(A)) + \dim(\cancel{col}(A)) = 3$ (Rank Theorem)

$\dim(null(A)) = 3 - \dim(row(A))$

$= 3 - 2 = 1$

(b) (i) $row(A) = span\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\right\} = span\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\} = row(rref(A))$ T

(ii) $col(A) = \mathbb{R}^2 = col(rref(A))$ T

(iii) $null(A) = span\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\} = null(rref(A))$ T

Note for ANY matrix A only (i) & (iii) are ALWAYS true, (ii) is not always true.

(c) Yes $\begin{pmatrix}3\\0\\-3\end{pmatrix}$ is in the row space of A.

$3\begin{pmatrix}1\\0\\-1\end{pmatrix} + 0\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}3\\0\\-3\end{pmatrix}$

(d) Every vector in $\mathbb{R}^2$ is in col(A), so <u>yes</u> $\begin{pmatrix}783\\-95\end{pmatrix} \in col(A)$.

2 $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2-4 \\ 2+2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$A\vec{x} = \lambda\vec{x}$

so $\lambda = -2$ and $\vec{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

3. PROVE If A is invertible THEN $\det(A) \neq 0$.

Prove the CONTRAPOSITIVE; IF $\det(A) = 0$ THEN A IS NOT invertible.

If $\det(A) = 0$ then $\lambda = 0$ is an eigenvalue

Since $\lambda = 0$ is an eigenvalue $\text{null}(A) = \text{null}(A - \lambda I)$ which is non-empty, which means A is NOT INVERTIBLE since $A\vec{x} = \vec{0}$ (i.e. null(A)) has more than 1 sol^n.

4. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} y \\ x \\ x+y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(a) T is a linear transformation ~~becomes~~ since

$T(c\vec{v}_1 + d\vec{v}_2) = cT(\vec{v}_1) + dT(\vec{v}_2)$

(b) (ii) $\dim(\text{range}(T)) = 2 = \dim(\text{col}([T]))$ is TRUE

5. $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(a) $\vec{u} \cdot \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1 - 1 + 0 = 0$

Yes, $\{\vec{u}, \vec{v}\}$ is an orthogonal set.

(b) $W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$. Since $\vec{u}$ and $\vec{v}$ are orthogonal they are also linearly independent, so they are a basis for W.

(c) A basis for $W^\perp$ will be a single vector which is orthogonal to both $\vec{u}$ and $\vec{v}$. The best way to do this is to find $\vec{u} \times \vec{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}\,1 - \hat{j}\,1 + \hat{k}(-2) = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

The other way is to find the basis of the left nullspace of A where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\text{rref}(A^T) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{pmatrix}$$

$$\text{null}(A^T) = \text{span}\begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right\}$$

Since the left null space is orthogonal to the column space, this basis will be orthogonal to $W = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$

5(d) $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\vec{w}^{\perp} = \text{proj}_{W^{\perp}}(\vec{b}) = \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$= \frac{2}{1+1+4} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\vec{w} = \vec{b} - \vec{w}^{\perp} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

orthogonal decomposition

$$\vec{b} = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

The long way to find $\vec{w}$ would be

$$\vec{w} = \text{proj}_{W}(\vec{b}) = \text{proj}_{\vec{u}}(\vec{b}) + \text{proj}_{\vec{v}}(\vec{b})$$

[since $W = \text{span}(\vec{u}, \vec{v})$]
$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$= \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{w} = 0 + \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$ which is the same as before

$$\vec{w}^{\perp} = \vec{b} - \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ 2/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$