Test 2: Linear Systems

Math 214 Spring 2008 Friday April 18
©Prof. Ron Buckmire 9:30pm-10:25pm

Name: ________________________________

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		30
2		40
3		30
BONUS		10
Total		**100**

1. Invertible Matrices. *30 points.*

Suppose A is an **invertible** $n \times n$ matrix, where $n > 1$.

(a) (*20 points.*) Determine whether each of the following is true or false. Just write T or F in front of each, without explanation.

________ The equation $A\vec{x} = \vec{0}$ has a nonzero solution.

________ For every vector $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ always has exactly one solution.

________ The rows of A form a basis for $\mathbb{R}^n$.

________ The null space of A has dimension n.

________ (reduced row echelon form) `rref`(A) is the identity matrix.

________ There exists a nonzero square matrix B such that $AB = \mathcal{O}$, where $\mathcal{O}$ is the square zero matrix.

________ A^{-1} is an invertible matrix.

________ A^T is an invertible matrix.

________ There exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

________ One of the eigenvalues of A must be zero.

(b) (*5 points.*) Choose one of the statements in part (a) that you determined is TRUE and **prove** that statement is true.

(c) (*5 points.*) Choose one of the statements in part (a) that you determined is FALSE and **prove** that statement is false.

2. Subspaces, Orthogonal Complements. *40 points.*

(a) (*12 points.*) Suppose that A is an $m \times n$ matrix and $\mathcal{V} = \left\{ \text{The set of vectors } \vec{y} \text{ such that } \vec{y}A = \vec{0} \right\}$. Is $\mathcal{V}$ a subspace of a vector space? PROVE YOUR ANSWER. (NOTE: You do not have to prove $\mathcal{V}$ is a vector space, just a subspace.)

(b) (*12 points.*) Suppose $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 1 \\ -3 & 17 & -20 & -2 \end{bmatrix}$. What is the rank of A? Use this information to write down the dimensions of all the four associated subspaces of the matrix A. EXPLAIN YOUR ANSWER.

(c) (*12 points.*) Again consider $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -2 & 1 \\ -3 & 17 & -20 & -2 \end{bmatrix}$. Find bases for $\mathcal{V}$ and its orthogonal complement $\mathcal{V}^{\perp}$ as defined in part (a). **SHOW ALL YOUR WORK**.

(d) (*4 points.*) Use your bases for $\mathcal{V}$ and $\mathcal{V}^{\perp}$ to verify they are indeed orthogonal complements.

3. Eigenvalues, Determinants, Diagonalization, Gram-Schmidt Orthogonalization, Orthogonal Matrices, Bases. *30 points.*

Do only one of the following problems:

Let $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$. Find A^{100}. SHOW ALL WORK. (HINT: diagonalize the matrix A; your answer should be a single 2×2 matrix).

OR

Find an orthogonal matrix Q which has the vector $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ as one of its columns.

SHOW ALL WORK. (HINT: use the given vector plus two more of your choice to produce an orthonormal basis for $\mathbb{R}^3$. To ease your calculations, choose vectors that have at least one zero component.) **Verify** the matrix you find is indeed orthogonal by computing QQ^T.

BONUS QUESTION. (*10 points.*)
Consider $\mathcal{V}$ and $\mathcal{V}^{\perp}$ from Question 2. Can the vector $(2, 1, 1)$ be expressed as a sum of two non-zero vectors $\vec{v} \in \mathcal{V}$ and $\vec{v}^{\perp} \in \mathcal{V}^{\perp}$? Find the values of $\vec{v}$ and $\vec{v}^{\perp}$ or explain why they do not exist.

OR

Prove that IF λ is an eigenvalue of a square matrix A, THEN $\det(A - \lambda\mathcal{I}) = 0$.