1. [20 points] Find the following determinants. The following information may (or may not) be helpful:

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5 \qquad \det(A) = 2 \qquad \det(BA^{-1}) = -6 \qquad \det(C) = 3$$

(a) $\det\begin{bmatrix} 0 & 2 & 0 & 0 \\ a & a & b & c \\ d & f & e & f \\ g & h & h & i \end{bmatrix} = 2\cdot(-1)^{1+2}\det\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2\cdot -1\cdot 5 = \boxed{-10}$

(b) $\det\begin{bmatrix} 2a-2d & 2b-2e & 2c-2f \\ -2d & -2e & -2f \\ 2g & 2h & 2i \end{bmatrix} = \begin{vmatrix} 2a & 2b & 2c \\ -2d & -2e & -2f \\ 2g & 2h & 2i \end{vmatrix} = -1\cdot\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix}$

$= -1\cdot 2^3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= -1\cdot 2^3\cdot 5$

$= \boxed{-40}$

(c) $\det\begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix} = x^3+y^3+z^3-xyz-xyz-xyz$

$= \boxed{x^3+y^3+z^3-3xyz}$

(d) $\det(B^TC^2) = \det(B^T)\det(C^2) = \det(B)(\det(C))^2$

$= \det(A)\det(BA^{-1})(3)^2$

$= 2\cdot 6\cdot 9$

$= \boxed{108}$

2. [20 points] $\lambda_1 = 0$ and $\lambda_2 = -2$ are the only eigenvalues of $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. For each eigenvalue, find a basis for the corresponding eigenspace.

$E_0 = \text{null}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x - z = 0 \Rightarrow x = z$

y free

$\vec{x} = \begin{pmatrix} x \\ y \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$E_{-2} = \text{null}(A + 2I)$

$\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$x = -z$

$y = 3z$

z free

$\vec{x} = \begin{pmatrix} -z \\ 3z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

$E_{-2} = \text{span}\left\{ \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right\}$

Is A invertible? Why or why not?

$\det(A) = 0 \cdot 0 \cdot -2 = 0$

Thus A^{-1} does NOT exist.

Is A diagonalizable? Why or why not?

A IS diagonalizable since the geometric multiplicity = algebraic multiplicity.

3. [10 points] Put an "X" mark in the blank prior to all the statements below that are true for $n \times n$ orthogonal matrices Q. Do not mark those statements that are not true for all $n \times n$ orthogonal matrices.

__X__ (a) $Q^T Q = I_n$

__X__ (b) $Q^T = Q^{-1}$

__X__ (c) $\texttt{dim}(\texttt{null}(Q)) = 0$

______ (d) Q has n distinct eigenvalues

______ (e) $\lambda = 1$ is an eigenvalue of Q

__X__ (f) $\texttt{det}(Q) = \pm 1$

__X__ (g) Q^{-1} is an orthogonal matrix

__X__ (h) $\|Q\vec{e}_1\| = 1$ where $\vec{e}_1$ is the usual standard unit vector in $\mathbf{R}^n$

__X__ (i) The columns of Q span $\mathbf{R}^n$

______ (j) $Q\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbf{R}^n$

HAPPY THANKSGIVING! Oh, wait, there's more test to go ...

4. [20 points] Mark each of the following "TRUE" or "FALSE". If I can't tell whether you wrote "T" or "F", you will not get credit for your answer.

T (a) If A is invertible, then $\mathtt{det}(A^{-1}) = \mathtt{det}(A^T)$.

F (b) If the $n \times n$ matrix A is diagonalizable, then it has n distinct (different) eigenvalues.

F (c) The geometric multiplicity of an eigenvalue is always greater than or equal to its algebraic multiplicity.

T (d) The matrix transformation $A\vec{x} = \begin{bmatrix} 2 & -2 & 0 & 1 \\ 1 & 5 & -2 & 0 \end{bmatrix} \vec{x}$ is a linear transformation.

F (e) All the eigenvectors of A are in the nullspace of A.

T (f) $\mathtt{det}(AB) = \mathtt{det}(BA)$

T (g) If $\mathtt{det}(A) \neq 0$, then A is invertible.

T (h) If $\lambda = 0$ is an eigenvalue of A, then A is not invertible.

F (i) If $\lambda = 0$ is an eigenvalue of A, then A is not diagonalizable.

F (j) A and B are similar if and only if B is a diagonal matrix, P is made up of the eigenvectors of A in its columns, and $P^{-1}AP = B$.

I know there's more still ... But at least there's no homework for Math 214 over break!

5. [10 points] Prove **ONE** of the following statements. Circle the one you are proving.

(a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x+2y \\ 3x-4y \end{bmatrix}$ is / is not a linear transformation. (Note: You must first decide if it is or is not a linear transformation, and then prove that.) [§3.6 #4]

(b) If A is an invertible matrix with eigenvalue λ and corresponding eigenvector $\vec{x}$, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$. [§4.3 # 13]

(c) If A is diagonalizable, then so is A^T. [§4.4 # 41]

(d) If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$. [§5.1 # 24]

(a) $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\vec{x}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \\ 3 & -4 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

T is a linear transformation, since it looks like $A\vec{x}$.

(b) $A\vec{x} = \lambda\vec{x} \Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$

Given $A\vec{x} = \lambda\vec{x}$ and A^{-1} exists

$$A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x})$$
$$I\vec{x} = \lambda A^{-1}\vec{x}$$
$$\vec{x} = \lambda A^{-1}\vec{x}$$
$$\vec{x} = A\vec{x}$$
$$\frac{1}{\lambda}\vec{x} =$$

(c) If A is diagonalizable, then so is A^T

$$AS = S\Lambda$$
$$\Rightarrow A = S\Lambda S^{-1}$$
$$A^T = (S\Lambda S^{-1})^T = (S^{-1})^T \Lambda^T S^T$$
$$= (S^T)^{-1} \Lambda S^T = R\Lambda R^{-1}$$

$\Lambda^T = \Lambda$ since Λ is a diagonal matrix

Let $R = (S^T)^{-1} \Rightarrow R^{-1} = S^T$ then $A^T = R\Lambda R^{-1}$

so A^T is diagonalizable.

6. [20 points] The following is one possible factorization of the matrix A:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = PDP^{-1}$$

$$= QDQ^T$$

(a) Show that the matrix $\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is orthogonal.

$$\begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{2}{5} - \frac{2}{5} = 0$$

$$\begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} = \frac{1}{5} + \frac{4}{5} = 1 \qquad \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = 4/5 + 1/5 = 1$$

(b) Why is it obvious that $\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$?

Since $Q^T = Q^{-1}$

(d) Write down the eigenvalue(s) for A? Next to each eigenvalue, write a basis for the corresponding eigenspace.

$\lambda = -3 \qquad E_{-3} = \text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

$\lambda = 2 \qquad E_2 = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$

(e) Explain how this factorization might help you find A^{10} more quickly than actually multiplying A together 10 times. [You don't actually have to do the calculation, but an appropriate equation and a complete explanation is needed for full credit.]

$$A^{10} = QD^{10}Q^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} (-3)^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

(BONUS) Why might we call matrix A "orthogonally diagonalizable"?

Because the matrices P are orthogonal matrices.