Report on Test 1

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Point Distribution (N=26)

Range	100+	90+	85+	79+	72+	68+	65+	60+	55+	50+	45+	40+	40-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	1	1	2	5	6	2	1	2	2	2	0	1	1

Summary Overall class performance was pretty good. The mean score was 71. The median score was 75. The high score was 101.

#1 Matrix Operations, Inverses, Transposes, Linear Combinations, Projections. These are TRUE or FALSE questions. **(a)** "There is no other matrix A besides the identity matrix $\mathcal{I}$ for which $AA^T = \mathcal{I}$." **FALSE.** $A\heartsuit = \mathcal{I}$ means that $\heartsuit$ must be equal to A^{-1}. Therefore you must find a matrix (other than the identity) for which $\heartsuit = A^T = A^{-1}$. All permutation matrices have this property. Simplest is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** "If $\vec{p}$ and $\vec{q}$ are solutions to a non-homogeneous linear system $A\vec{x} = \vec{b}$, then so is any linear combination of $\vec{p}$ and $\vec{q}$." **FALSE.** This was Clicker Question 10.3. It is true that if a linear system has two solutions then you know that it must have an infinite number of solutions. However, infinite does not mean that EVERY linear combination $c_1\vec{p} + c_2\vec{q}$ of $\vec{p}$ and $\vec{q}$ will solve the system. It turns out that only linear combinations such that $c_1 + c_2 = 1$ will also solve the system. **(c)** "For every non-zero vector $\vec{u}$ and $\vec{v}$ in R^n, $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ and $\vec{v}$ are orthogonal." **TRUE.** Drawing a picture does make it clear that the statement appears to be true in R^2 for two random vectors but this is not a proof. A proof is using the definition, which requires using the dot product. $\vec{v} \cdot (\vec{u} - \text{proj}_{\vec{v}}(\vec{u})) = \vec{v} \cdot \vec{u} - \vec{v} \cdot \left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \right) = \vec{v} \cdot \vec{u} - \left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \cdot \vec{v} = \vec{v} \cdot \vec{u} - \vec{u} \cdot \vec{v} = 0.$
(d) "If $A^3 = \mathcal{O}$, then $(\mathcal{I} - A)^{-1} = \mathcal{I} + A + A^2$." **TRUE.** Recalling $P^3 - Q^3 = (P - Q)(P^2 + PQ + Q^2)$, multiply both sides of the equation by $(\mathcal{I} - A)$ to produce $(\mathcal{I} - A)^{-1}(\mathcal{I} - A) = (\mathcal{I} + A + A^2)(\mathcal{I} - A) \Rightarrow \mathcal{I} = \mathcal{I} + A + A^2 - A - A^2 - A^3 = \mathcal{I}^3 - A^3$. Since $\mathcal{I}^3 = \mathcal{I}$ the initial statement will be true when $A^3 = \mathcal{O}$.

#2 Solution Sets, Equations of Planes and lines, Homogeneous Systems. This question is almost identical to BONUS Quiz #2. It is an opportunity to illustrate your ability to obtain solution sets of linear systems and interpret the results. **(a)** By doing elementary row operations, it turns out that $x+z = 0, y+z = 0$ and z is free (so $z = t$). This corresponds to $\vec{x} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ which is a line through the origin. **(b)** By doing elementary row operations on the second (non-homogeneous) system, it turns out that $x+z = 2, y+z = -1$ and z is free (so $z = t$). This set corresponds to $\vec{x} = \begin{bmatrix} 2-z \\ -1-z \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, which is a line in R^3 not through the origin. **(c)** The two solutions to the linear systems correspond to parallel lines, since the direction vectors are the same.

#3 Rank, Span, Invertibility, Reduced Row Echelon Form. This question is about showing your ability to obtain Reduced Row Echelon Form and show that you can obtain information from that result. **(a)**. The rnk is the number of non-zero rows, which is 2. Because the rank is 2 and the vectors are all in $\mathbb{R}^2$ the span is *every* vector$\mathbb{R}^2$. The vectors are **linearly dependent** because there are 3 of them in $\mathbb{R}^2$ and the Rank Theorem says there will be a free variable in the homgeneous solution. M is not invertible because it's not square. **(b)** This time the rank is 3. Since this means $\texttt{rref}(M) = \mathcal{I}$ you know the column vectors are **linearly independent** and M is invertible. Since they are 3 linearly independent vectors in $\mathbb{R}^3$ the span is all of $\mathbb{R}^3$. **(c)**. The rank is again 3 and since there are 3 vectors they all must be linearly independent. Just because there is a zero row does NOT mean the colum vectors are linearly dependent! $M\vec{x} = \vec{0}$ has one solution, so they columns are linearly independent. M is not invertible because it's not square.

#4 Orthogonality, Dot Product, Homogeneous Systems. (a) For 2 vectors to be orthogonal, their dot product must equal zero. The conditions $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$ corresponds to $a = 0, a+b+c+d = 0, 2a+b+d = 0$. **(b)** The solution set turns out to be $a = 0, b = -d, c = 0, d$free which corresponds to $\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$. **(c)** $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0.$

BONUS `span`$(\mathcal{P})$ from Question 4 is a hyperplane in $\mathbb{R}^4$; it has dimension 3. You know the object has dimension 3 because it's span contains 3 linearly independent vectors in $\mathbb{R}^4$. The vector form of the equation to describe it is $\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ where c_i are any real number. Using the result from Question 4 that the normal to `span`$(\mathcal{P})$ is $\vec{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ means that one can obtain the general equation, which is $\vec{n} \cdot \vec{x} = 0$ where $\vec{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ so that $x - z = 0$ or $x = z$. The normal form is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.