1. Matrix Operations, Inverses, Transposes, Linear Combinations, Projections. *24 points.*

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember, if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

NOTE: In the questions below $\mathcal{I}$ is the $n \times n$ identity matrix and $\mathcal{O}$ is the $n \times n$ zero matrix.

(a) *6 points.* **TRUE or FALSE?** "There is no other matrix A besides the identity matrix $\mathcal{I}$ for which $AA^T = \mathcal{I}$."

$\boxed{\text{FALSE}}$

Any matrix that has $A^T = A^{-1}$ will satisfy this equation. These are permutation matrices such as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(b) *6 points.* **TRUE or FALSE?** "If $\vec{p}$ and $\vec{q}$ are solutions to a non-homogeneous linear system $A\vec{x} = \vec{b}$, then so is any linear combination of $\vec{p}$ and $\vec{q}$."

$\boxed{\text{FALSE}}$

$A\vec{p} = \vec{b}$ and $A\vec{q} = \vec{b}$

But $A(c_1\vec{p} + c_2\vec{q}) = c_1 A\vec{p} + c_2 A\vec{q}$

$= c_1 \vec{b} + c_2 \vec{b}$

$= (c_1 + c_2)\vec{b} = \vec{b}$ only if $c_1 + c_2 = 1$

The statement is true only ~~when~~ for SOME linear combs of $\vec{p}$ and $\vec{q}$.

(c) *6 points.* **TRUE or FALSE?** "For every non-zero vector $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ and $\vec{v}$ are orthogonal."

$\boxed{\text{TRUE}}$

$$\vec{v} \cdot \left(\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} \right) \vec{v} \right) = \vec{v}\cdot\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}} \right) \vec{v}\cdot\vec{v}$$

$$= \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v}$$

$$= 0$$

(d) *6 points.* **TRUE or FALSE?** "If $A^3 = \mathcal{O}$, then $(\mathcal{I} - A)^{-1} = \mathcal{I} + A + A^2$."

$\boxed{\text{TRUE}}$

$$(I - A)^{-1}(I - A) = (I + A + A^2)(I - A)$$

$$I = I + A + A^2 - A - A^2 - A^3$$

$$= I - A^3$$

If $A^3 = \mathcal{O}$ then $I = I$, so

$(I - A)^{-1} = I + A + A^2$

2. Solution Sets, Equations of Planes and Lines, Homogeneous Systems. *26 points.*

For each of the linear systems below: (i) Find the solution set of the linear system and (ii) Identify the geometrical object which represents the solution of the linear system and (iii) Write down the equation of the geometric object in vector form.

(a) (*10 points.*)

$$\begin{aligned} x + y + 2z &= 0 \\ y + z &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$x + z = 0$
$y + z = 0$
z free!

$\vec{x} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix}$, z anything

$\vec{x} = z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, $z \in \mathbb{R}$

This is a line thru the origin in $\mathbb{R}^3$.

(b) (*10 points.*)

$$\begin{aligned} x + y + 2z &= 1 \\ y + z &= -1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)$$

$x + z = 2$
$y + z = -1$.
z free!

$\vec{x} = \begin{pmatrix} 2 - z \\ -1 - z \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, z any

This is a line in $\mathbb{R}^3$ NOT thru origin.

(c) (*6 points.*) What is the (geometric) relationship between the solution sets to the above linear systems? **EXPLAIN YOUR ANSWER.**

The two solution sets represent ~~h~~ parallel lines in $\mathbb{R}^3$. The direction vectors are the same.

3. Rank, Span, Invertibility, Reduced Row Echelon Form. *30 points.*

(a) (*10 points.*) Suppose $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. By considering the vectors in $\mathcal{P}$ as the columns of a matrix M show that the reduced row echelon form of M, rref(M)$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find rank(M) and describe span($\mathcal{P}$). What can you say about this set of vectors: are they linearly independent, linearly dependent or not enough information is given? Also, is M invertible? **EXPLAIN YOUR ANSWERS.**

2pt rank(M) = 2 (# of non-zero rows)

2pt span(P) = $\mathbb{R}^2$ (every vector in $\mathbb{R}^2$) 2 lin ind vectors in $\mathbb{R}^2$

3pt $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1' = R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

1pts M is NOT INVERTIBLE since M is rectangular!

2pts The columns of M are LINEARLY DEPENDENT since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or There are infinite solutions to the problem as seen in 2(a).

(b) (*10 points.*) Suppose that $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. By considering the vectors in $\mathcal{P}$ as the columns of a matrix M show that the reduced row echelon form of M, rref(M)$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find rank(M) and describe span($\mathcal{P}$). What can you say about this set of vectors: are they linearly independent, linearly dependent or not enough information is given? Also, is M invertible? **EXPLAIN YOUR ANSWERS.**

2pt rank(M) = 3

2pt Span(P) = $\mathbb{R}^3$ (3 lin indep vectors in $\mathbb{R}^3$)

3pts $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2' = R_2 - R_3} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow[R_1' = R_1 - 2R_2]{R_1' = R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2pts The columns of M are linearly independent! The only solution to $M\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

1pt M <u>is</u> invertible (since $M\vec{x} = \vec{0}$ has a unique sol^n)

(c) (*10 points.*) Suppose that $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Show that by considering the vectors in $\mathcal{P}$ as the columns of a matrix M the reduced row echelon form of M, $\texttt{rref(M)}= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and find $\texttt{rank(M)}$. What can you say about this set of vectors: are they linearly independent, linearly dependent or not enough information is given? Also, is M invertible? **EXPLAIN YOUR ANSWERS.**

2pts rank(M) = 3

4pts
$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_2]{R_4' = R_4 - R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_1' = R_1 - R_2]{R_3' = R_3 - R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1' = R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

3pts The vectors are linearly independent since the only solution to $M\vec{x} = \vec{0}$ would be $\vec{x} = 0$.

1pt M is NOT invertible since M is NOT SQUARE

span(M) [NOT ASKED FOR] is a 3-D object in $\mathbb{R}^4$.

4. Orthogonality, Dot Product, Homogeneous Systems. *20 points.*

We want to find out if it is possible to obtain a vector $\vec{n} = \begin{bmatrix} a, & b, & c, & d \end{bmatrix}$ which is orthogonal to every vector in span($\mathcal{P}$) where $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) (*6 points.*) Write down three linear equations in the variables a, b, c and d that describe the condition that a vector $\vec{n}$ is orthogonal to each of the vectors in $\mathcal{P}$.

3pts

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$a = 0$$
$$a + b + c + d = 0$$
$$2a + b + d = 0$$

3pts

(b) (*9 points.*) Find the solution set of the linear system you wrote down in part (a).

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3' = R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3' = -1R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2' = R_2 - R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

4pts

$$a = 0$$
$$b + d = 0$$
$$c = 0$$

$$\vec{x} = \begin{pmatrix} 0 \\ -d \\ 0 \\ d \end{pmatrix}, \ d \text{ any}$$

5pts

(c) (*5 points.*) Use your answer in (b) to write down a vector $\vec{n}$ that is indeed orthogonal to every vector in $\mathcal{P}$ and confirm this property.

2pts $\vec{n} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 + 0 + 0 + 0 = 0 \checkmark \qquad 1pt$$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0 - 1 + 0 + 1 = 0 \checkmark \qquad 1pt$$

1pt $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0 - 1 + 0 + 1 = 0 \checkmark$

BONUS QUESTION. Analytic Geometry, Vector Form, Normal Form, General Form, Dimension. (*10 points.*)

Consider the span of $\mathcal{P}$ from the two previous problems as a geometric object. Using information from Question 3 and Question 4 you should be able to write down the normal form, the vector form and the general form of the equation of the geometric object represented by span($\mathcal{P}$). [HINT: some of these are easier to write down than others!] What is the dimension of this object and what is the name of this geometric object?

VECTOR FORM

$$\text{Span}(P) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + w\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad s, t, w \in \mathbb{R}$$

NORMAL FORM

$$\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

GENERAL FORM

$$-x + z = 0$$

Span(P) is a 3-D object in $\mathbb{R}^4$ (1 eq^n to represent it)

Span(P) is a HYPERPLANE!