Test 1: LINEAR SYSTEMS

Math 214 Spring 2008   Friday February 29
©Prof. Ron Buckmire   9:30pm-10:25pm

Name: _____

Directions: Read *all* problems first before answering any of them. There are 7 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, _____, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		24
2		26
3		30
4		20
BONUS		10
Total		**100**

1. Matrix Operations, Inverses, Transposes, Linear Combinations, Projections.
24 points.

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember, if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

NOTE: In the questions below $\mathcal{I}$ is the $n \times n$ identity matrix and $\mathcal{O}$ is the $n \times n$ zero matrix.

(a) *6 points.* **TRUE or FALSE?** "There is no other matrix A besides the identity matrix $\mathcal{I}$ for which $AA^T = \mathcal{I}$."

$$\boxed{}$$

(b) *6 points.* **TRUE or FALSE?** "If $\vec{p}$ and $\vec{q}$ are solutions to a non-homogeneous linear system $A\vec{x} = \vec{b}$, then so is any linear combination of $\vec{p}$ and $\vec{q}$."

$$\boxed{}$$

(c) *6 points.* **TRUE or FALSE?** "For every non-zero vector $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ and $\vec{v}$ are orthogonal."

$$\boxed{}$$

(d) *6 points.* **TRUE or FALSE?** "If $A^3 = \mathcal{O}$, then $(\mathcal{I} - A)^{-1} = \mathcal{I} + A + A^2$."

$$\boxed{}$$

2. Solution Sets, Equations of Planes and Lines, Homogeneous Systems. *26 points.*

For each of the linear systems below: (i) Find the solution set of the linear system and (ii) Identify the geometrical object which represents the solution of the linear system and (iii) Write down the equation of the geometric object in vector form.

(a) (*10 points.*)

$$\begin{aligned} x + y + 2z &= 0 \\ y + z &= 0 \end{aligned}$$

(b) (*10 points.*)

$$\begin{aligned} x + y + 2z &= 1 \\ y + z &= -1 \end{aligned}$$

(c) (*6 points.*) What is the (geometric) relationship between the solution sets to the above linear systems? **EXPLAIN YOUR ANSWER**.

3. Rank, Span, Invertibility, Reduced Row Echelon Form. *30 points.*

(a) (*10 points.*) Suppose $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. By considering the vectors in $\mathcal{P}$ as the

columns of a matrix M show that the reduced row echelon form of M, $\texttt{rref}(M) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Find $\texttt{rank}(M)$ and describe $\texttt{span}(\mathcal{P})$. What can you say about this set of vectors: are they linearly independent, linearly dependent or not enough information is given? Also, is M invertible? **EXPLAIN YOUR ANSWERS**.

(b) (*10 points.*) Suppose that $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. By considering the vec-

tors in $\mathcal{P}$ as the columns of a matrix M show that the reduced row echelon form of M,

$\texttt{rref}(M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find $\texttt{rank}(M)$ and describe $\texttt{span}(\mathcal{P})$. What can you say about this

set of vectors: are they linearly independent, linearly dependent or not enough information is given? Also, is M invertible? **EXPLAIN YOUR ANSWERS**.

(c) (*10 points.*) Suppose that $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Show that by consider-

ing the vectors in $\mathcal{P}$ as the columns of a matrix M the reduced row echelon form of M,

$\mathtt{rref}(M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and find $\mathtt{rank}(M)$. What can you say about this set of vectors: are

they linearly independent, linearly dependent or not enough information is given? Also, is M invertible? **EXPLAIN YOUR ANSWERS**.

4. Orthogonality, Dot Product, Homogeneous Systems. *20 points.*

We want to find out if it is possible to obtain a vector $\vec{n} = \begin{bmatrix} a, & b, & c, & d \end{bmatrix}$ which is orthogonal to every vector in $\mathtt{span}(\mathcal{P})$ where $\mathcal{P} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) (*6 points.*) Write down three linear equations in the variables a, b, c and d that describe the condition that a vector $\vec{n}$ is orthogonal to each of the vectors in $\mathcal{P}$.

(b) (*9 points.*) Find the solution set of the linear system you wrote down in part (a).

(c) (*5 points.*) Use your answer in (b) to write down a vector $\vec{n}$ that is indeed orthogonal to every vector in $\mathcal{P}$ and confirm this property.

BONUS QUESTION. Analytic Geometry, Vector Form, Normal Form, General Form, Dimension. (*10 points.*)

Consider the span of $\mathcal{P}$ from the two previous problems as a geometric object. Using information from Question 3 and Question 4 you should be able to write down the normal form, the vector form and the general form of the equation of the geometric object represented by $\mathtt{span}(\mathcal{P})$. [HINT: some of these are easier to write down than others!] What is the dimension of this object and what is the name of this geometric object?