

Test 1: LINEAR SYSTEMS

Math 214 Spring 2007
©Prof. Ron Buckmire

Friday March 2
2:30pm-3:25pm

Name: R. BUCKMIRE

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and **CLEARLY** indicate your final answers to be graded from your "scratch work."

Pledge: I, _____, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		20
2		25
3		30
4		25
BONUS		10
Total		100

1. Span, Linear Independence, Rank. 20 points.

Consider the matrix $A = \begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix} \rightarrow \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(a) (4 points.) Find the reduced row echelon form of A , $\text{rref}(A)$.

$$\text{rref}(A) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

(b) (4 points.) With or without your knowledge of $\text{rref}(A)$, what is the rank of the matrix A ? **EXPLAIN YOUR ANSWER.**

$$\text{rank} = 1$$

Number of non-zero rows

(c) (4 points.) With or without your knowledge of $\text{rref}(A)$, what is the span of the columns of matrix A ? **EXPLAIN YOUR ANSWER.**

$$\text{span}\left\{\begin{pmatrix} 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 6 \\ 18 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\} = \text{the line } y = 3x$$

(d) (4 points.) With or without your knowledge of $\text{rref}(A)$, discuss the linear independence of the columns of the matrix A . **EXPLAIN YOUR ANSWER.**

$$\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 6 \\ 18 \end{pmatrix} \quad \text{The columns are linear dependent}$$

(e) (4 points.) With or without your knowledge of $\text{rref}(A)$, discuss whether the matrix A^{-1} exists. **EXPLAIN YOUR ANSWER.**

$$A^{-1} \text{ does not exist since } \text{rref}(A) \neq I$$

2. Row reduction, Reduced Row Echelon Form, Identity, Invertibility. 25 points.

(a) (8 points.) Show that $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & 2 & 6 \end{bmatrix}$ can be transformed into $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ using elementary row operations.

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & 2 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (8 points.) Show that I_3 can be transformed into $B = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ using elementary row operations.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) (9 points.) Is it possible to transform A into B using elementary row operations? Explain why you can not or explain why you can, without going through the actual work of proving the result. Is A invertible? Is B invertible? How do you know? EXPLAIN YOUR ANSWERS.

$$A \rightarrow I_3 \rightarrow B$$

Yes it's possible.

$$\begin{aligned} (E_k \dots E_1) A &= I_3 \\ A &= (E_k \dots E_1)^{-1} \\ A^{-1} &= (E_k \dots E_1) \end{aligned}$$

$$\begin{aligned} EI &= B \\ I &= E^{-1}B \end{aligned} \quad E^{-1} = B^{-1}$$

3. Matrix Operations, Trace, Transpose. 30 points.

TRUE or FALSE – put your answer in the box (1 point). To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! Remember if you think a statement is TRUE you must prove it is ALWAYS true. If you think a statement is FALSE then all you have to do is show there exists a counterexample which proves the statement is FALSE at least once.

The trace of a $n \times n$ matrix A (where A_{ij} is the element in the i^{th} row and j^{th} column) is defined as the sum of the diagonal elements of A and denoted $\text{tr}(A)$. In other words,

$$\text{tr}(A) = \sum_{i=1}^n A_{ii}$$

(a) 10 points. TRUE or FALSE? “The trace of the $n \times n$ identity matrix I_n , $\text{tr}(I_n)$ equals n .”

TRUE

$$\text{tr}(I_n) = \sum_{i=1}^n 1 = n$$

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

(b) 10 points. TRUE or FALSE? “ $\text{tr}(A)\text{tr}(B) = \text{tr}(AB)$ for every $n \times n$ matrix A and B .”

FALSE

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$$\begin{aligned} \text{tr} A &= 5 \\ \text{tr} B &= 13 \\ \text{tr} AB &= 65 \end{aligned}$$

$$\text{tr}(AB) = 69$$

(d) 10 points. TRUE or FALSE? “ $\text{tr}(AA^T)$ equals the sum of the square of each element in A for every $n \times n$ matrix A .”

TRUE

$$\begin{aligned} (AA^T)_{ii} &= \text{row}_i(A) \cdot \text{col}_i(A^T) \\ &= \text{row}_i(A) \cdot \text{row}_i(A) = \sum_{j=1}^n a_{ij}^2 \\ &= |\text{row}_i(A)|^2 \end{aligned}$$

$$\text{tr}(AA^T) = \sum_{i=1}^n (AA^T)_{ii} = \sum_{i=1}^n |\text{row}_i(A)|^2 = \sum_{i=1}^n (A_{i1}^2 + A_{i2}^2 + \dots + A_{in}^2)$$

4. Linear Systems, Equations of Planes and Lines. 23 points.

Consider the linear system

$$\begin{aligned} 3x + 5y - 4z &= 0 \\ -3x - 2y + 4z &= 0 \\ 6x + y - 8z &= 0 \end{aligned}$$

(a) (10 points.) Find the non-trivial solution(s) of the linear system.

$$\begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & -9 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} 3x - 4z &= 0 \\ y &= 0 \end{aligned} \Rightarrow \begin{aligned} x &= \frac{4z}{3} \\ x &= t \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \end{aligned} \text{ for any } t \in \mathbb{R}$$

(b) (10 points.) What is the geometrical object which represents the solution of the linear system? Write down its equation in vector form.

The object is a line, The intersection of 3 planes through the origin.

In vector form the equation of the line is

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

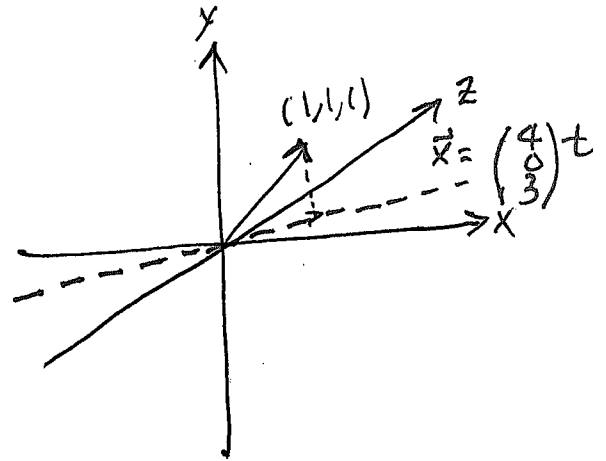
(c) (5 points.) What is the smallest distance between the geometrical object which represents the solution of the linear system and the origin (0, 0, 0). EXPLAIN YOUR ANSWER.

Since the line goes through the origin, the distance is zero.

BONUS QUESTION. Analytic Geometry, Projections. (10 points.)

Find the smallest distance between the geometrical object which represents the solution to the linear system in Question 4 and the point $(1, 1, 1)$.

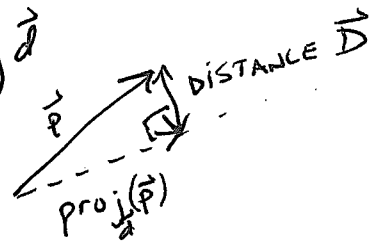
The point $(0, 0, 0)$ is on the line.



We need to find projection of $\vec{p} = (1, 1, 1)$ in the \vec{d} direction $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

$$\text{DISTANCE} + \text{proj}_{\vec{d}}(\vec{p}) = \vec{p}$$

$$\text{DISTANCE } \vec{D} = \vec{p} - \text{proj}_{\vec{d}}(\vec{p}) = \vec{p} - \left(\frac{\vec{p} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \right) \vec{d}$$



$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}}{\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{7}{25} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 28/25 \\ 0 \\ 21/25 \end{pmatrix} = \begin{pmatrix} 3/25 \\ 1 \\ 4/25 \end{pmatrix}$$

$$\boxed{\frac{\sqrt{26}}{5} = |\vec{D}|}$$

$$= \sqrt{\left(\frac{3}{25}\right)^2 + \left(\frac{4}{25}\right)^2 + 1^2} = \frac{1}{25} \sqrt{3^2 + 4^2 + 25^2}$$

$$= \frac{\sqrt{26 \cdot 25}}{25} = \frac{\sqrt{26} \cdot \sqrt{25}}{25} = \frac{\sqrt{26} \cdot 5}{25} = \frac{\sqrt{26}}{5}$$

The line is within $\frac{\sqrt{26}}{5}$ th unit of the point $(1, 1, 1)$ at the closest point.

(confirm \vec{D} is orthogonal to \vec{p} so \vec{D} is minimum distance)

$$\begin{aligned} \vec{D} \cdot \text{proj}_{\vec{d}}(\vec{p}) &= \left(\vec{p} - \left(\frac{\vec{p} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \right) \vec{d} \right) \cdot \left(\frac{\vec{p} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \right) \vec{d} \\ &= \frac{(\vec{p} \cdot \vec{d})^2}{\vec{d} \cdot \vec{d}} - (\vec{p} \cdot \vec{d}) \frac{(\vec{p} \cdot \vec{d})}{(\vec{d} \cdot \vec{d})} \\ &= 0 \end{aligned}$$